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UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER –II

POINT SET TOPOLOGY

DEMATH-2 CORE-2

BLOCK-2

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

POINT SET TOPOLOGY

BLOCK 1

BLOCK 2

BLOCK-2 POINT SET TOPOLOGY

Introduction To The Block

Unit 8 Countability and Separation Axioms-I: Deals with Sepration Axioms and Countability

Unit 9 Countability and Separation Axioms-II: Deals with Regular and Normal Topological Space. Urysohn Lemma, Tietze Extension Theorem, Urysohn Metrization Theorem

Unit 10 Second Countable Space : Deals Equivalent Condition for a Space to be Tychonoff and Metrization Theorem

Unit 11 Connected And Path-Connected Space : Deals with Connected and Path-Connected Space also deals with connected Components

Unit 12 Compact Space : Deals with compact space and Lindelof Property

Unit 13 Different Kind Of Compactness : Deals with Local compactness and Stone–Čech Compactification

Unit 14 Covering Space and Uniform Space : Deals with covering space and its properties and total boundedness

UNIT 8 : COUNTABILITY AND SEPARATION AXIOMS-I

STRUCTURE

8.0 OBJECTIVE

- Learn countability and separation Axiom
- Learn Regular and Normal Topological Space
- Learn Invarient , Orientability and Mobius
- Learn Genus and Euler Characteristic

8.1 INTRODUCTION.

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms. These are sometimes called Tychonoff separation axioms, after Andrey Tychonoff.

The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of kinds of topological spaces. However, the term "separation axiom" has stuck. The separation axioms are denoted with the letter "T" after the German Trennungsaxiom, which means "separation axiom."

The precise meanings of the terms associated with the separation axioms has varied over time, as explained in History of the separation axioms. It is important to understand the authors' definition of each condition mentioned to know exactly what they mean, especially when reading older literature.

1.1 Countability Properties

There are two basic themes to the next several sections:

a. What properties of a topology allow us to conclude that the topology is given by a metric?

b. What properties of a space allow us to conclude that the space actually is (homeomorphic to) a subspace of R n (or at least a subspace of Rω)?

Countability Properties: Here are several properties of spaces, all saying that the topology, or some key feature of it, can be described in terms of countably many pieces of information. The names are historical; they are not very descriptive or otherwise useful, but you should know them since they are used in the literature.

(1) “First axiom of countability” The space (X, T) is called first-countable if the topology has a countable local basis at each point $x \in X$.

(2) ”Second axiom of countability” The space (X, T) is called second-countable if the topology T has a countable basis.

(3) ”Separable” The space (X, T) is called separable if X contains a countable dense subset. Recall a subset $A \subseteq X$ is called dense in X if the closure $\bar{A}$ is all of X, i.e. each open set contains at least one point of A.

(4) Lindel¨of property The space (X, T) is called a Lindel¨of space if each open cover of X has a countable sub-cover.

The familiar space R^n, with the standard topology has all of the above properties (proof below). For more general spaces, we can ask many questions:

- Do any of these properties imply others?
- If a space X has one of the properties, do all subspaces of X have the property? (In that always happens, we would call the property hereditary.)
- If we have a family of spaces with one of these properties, does the cartesian product have the property?
- If $f : X \to Y$ is a continuous surjection, and X has one of the properties, must Y also have the property?
- If (X, T) has one of these properties, and T' is a coarser [resp. finer] topology, must (X, T') have the property?

We will focus on just some highlights.

Theorem . Countable basis =⇒ all the other countability properties.

Proof. Suppose B is a countable basis for the topology on X.

a. Countable local basis: Let $x \in X$ and let U be any neighborhood of x. Since B is a basis for the topology, U is a union of elements of B. Thus there exists an element $B \in B$ such that $x \in B \subseteq U$. So the set B is a countable local basis for each point $x \in X$.

b. .Separable: For each nonempty set B ∈ B, pick a point xB ∈ B. Since B is countable, the set {xB | B ∈ B} is countable. Since each open set is a union of elements of B, each nonempty open set U contains at least one of the sets B and so xB ∈ U. Thus {xB | B ∈ B} is dense in X.

c. Lindel¨of : Let {Uα}α∈J be an open cover of X. We want to prove there exists a countable subcover, by somehow using the existence of a countable basis B for the topology. For convenience (to make the exact argument a little simpler), assume that one of the sets Uα is actually the empty set. (Or adjoin one additional set U0 = ∅ to the covering.) The idea of the proof is to use the elements of B to "point to" certain special Uα's. Specifically, for each set B ∈ B, we will select one set UB from among the Uα's as follows: First ask if there exists at least one of the open sets Uα containing that set B. If not, let UB = U0 = ∅. If the set B is contained in some Uα, then pick one such Uα and call it UB. We might pick the same Uα corresponding to several B's (because a given Uα usually contains many basis sets), but we have at most one Uα chosen for each B; so the set {UB : B ∈ B} is countable. We now show that {UB : B ∈ B} covers X. Let x ∈ X. We shall prove that at least one of the sets UB contains x. Since the Uα's cover X, there is some Uα containing x. Since B is a basis, there exists B ∈ B with x ∈ B ∈ Uα. Since that basis set B is contained in some Uα, B is one of the basis sets for which we chose a set UB ⊇ B. So x ∈ B ⊆ UB, in particular x ∈ UB.

The previous theorem says that having a countable basis for the topology is the strongest of the countability properties. The next example shows that it is strictly stronger, that is the other properties do not imply it.

Example. The space Rℓ is first-countable, separable, and Lindel¨of , but not second-countable.

Proof. The details are given in the text; you should be able to prove Rℓ is first-countable and separable, and that it is not second-countable. You are not required to know the proof that Rℓ is Lindel¨of .

The previous example shows some of the independence of the properties. However, in metric spaces, the first-countable, separable, and second-countable properties are equivalent.

Theorem . Suppose X is a metric space. Then

- **i. X has countable local bases at each point.**
- **ii. X separable =⇒ X has a countable basis.**
- **iii. X Lindel¨of =⇒ X has a countable basis.**

Proof. i. The idea of "countable local basis" is precisely a generalization of the balls of radius 1/n in metric spaces: The set {B(x, 1 n)} is a local basis at x

ii. Let {xn}n∈N be a countable dense set in X. For each xn, let Bn be the set of all open balls centered at xn with rational radius. Then the set Bn is countable for each n, so the set B = ∪{Bn : n ∈ N} is countable. We claim this set B is a basis. The proof is an exercise in using the triangle inequality that the metric satisfies. (You can work out the details: here is the idea...) Take any open set U ⊆ X. We want to show that U is a union of our alleged basis elements. Let y be any point of U; we shall show that there is one of our alleged basis sets B such that y ∈ B ⊆ U. The point y is contained in some ϙ ball inside U. Now look at an ϙ/100 ball around y. This ball must contain some point xn from our countable dense subset. So the distance from xn to y is less than ϙ/100. Then by picking a rational radius slightly larger than ϙ/100, we can find a rational-radius ball centered at xn, containing y, and contained in U.

iii For each n, consider the open covering of X consisting of all balls of radius 1/n. The Lindel¨of property says there exists a countable subcover Bn. Let B = ∪{Bn n ∈ N}. This is a countable union of countable sets, hence countable. Check that it is indeed a basis.

Remark. The the preceding proofs, it might seem that all we need is "separable + countable local basis" or "Lindel¨of + countable local basis" to conclude that X has a countable basis. But remember the previous example of a space that IS separable, DOES have the Lindel¨of property, DOES have a countable local basis at each point, but does not have a countable basis for the topology. The triangle inequality property of metrics provides an extra amount of “niceness” for the topology, so we can connect from one property to the other.

Now let us consider subspaces and cartesian products.

Theorem. The properties countable local basis and countable basis are preserved for subspaces and for countable cartesian products.

Proof. Proofs are given in the text.

Theorem. The Lindel¨of property is inherited by closed subspaces (analogous to compactness).

Example. The property of being separable need not be inherited by subspaces; having the subspace be closed seems irrelevant to this question.

The space Rℓ × Rℓ is separable but the diagonal line {(x, −x) : x ∈ R} is a closed subspace that is homeomorphic to R with the discrete topology. This uncountable closed discrete subspace makes R 2 ℓ a useful counterexample for several questions. It shows that being separable is not hereditary, the Lindel¨of property is not always preserved by finite products, and [next section] the property of being normal is not always preserved by finite products..

8.2 PROPERTIES OF FIRST COUNTABLE TOPOLOGICAL SPACES

Theorem. If (X, J) is a first countable topological space then for each x ∈ X there exists a countable local base say {Vn(x)} ∞ n=1 such that Vn+1(x) ⊆ Vn(x)

Proof. Fix x ∈ X. Now (X, J) is a first countable topological space implies there exists a countable local base say $\{Un\}_{n=1}^{\infty}$ at x. Let Vn(x) = U1 ∩ U2 ∩ · · · ∩ Un then $\{Vn(x)\}_{n=1}^{\infty}$ is a collection of open sets such that Vn+1(x) ⊆ Vn(x) for all n ∈ N. So, it is enough to prove that

$\left\{ Vn(x) \right\}_{n=1}^{\infty}$ is a local base at x. So start with an open set V containing x. Now $\left\{ Un \right\}_{n=1}^{\infty}$ is a local base at x and V is an open set containing x implies there exists n0 ∈ N such that Un0 ⊆ V . By the definition of Vn(x) 0 s we have Vn0 (x) ⊆ Un0 . Hence we have the following: for each open set V containing x there exists n0 ∈ N such that Vn0 (x) ⊆ V . This implies that {Vn(x)} is a local base at x satisfying Vn+1(x) ⊆ Vn(x) for all n ∈ N.

Let us use the above characterization of a first countable base to show that, in some sense, first countable topological spaces behave like metric spaces.

Theorem. Let (X, J) be a first countable topological space and A be a nonempty subset of X. Then for each x ∈ X, x ∈ A if and only if there exists a sequence $\left\{ xn \right\}_{n=1}^{\infty}$ in A such that xn → x as n → ∞.

Proof. First let us assume that x ∈ A. Now (X, J) is a first countable topological space implies there exists a countable local base say B = {Vn} ∞ n=1 such that Vn+1 ⊆ Vn, for all n ∈ N. Hence x ∈ A implies A ∩ Vn 6= φ, for each n ∈ N. Let xn ∈ A ∩ Vn. Claim: xn → x as n → ∞.

So start with an open set U containing x (enough to start with Vn) then there exists n0 ∈ N such that x ∈ Vn0 ⊆ U. Hence xn ∈ Vn ⊆ Vn0 ⊆ U for all n ≥ n0. That is xn ∈ U for all n ≥ n0. This means xn → x as n → ∞.

Conversely, suppose there exists a sequence {xn} ∞ n=1 in A such that xn → x. Then for each open set U containing x there exists a positive integer n0 such that xn ∈ U for all n ≥ n0. In particular xn0 ∈ U ∩ A. Hence for each open set U containing x, U ∩ A 6= φ and this implies x ∈ A

Theorem Let X and Y be topological spaces and further suppose X is a first countable topological space. Then a function f : X → Y is continuous at a point x ∈ X if and only if for every sequence {xn} ∞

n=1 in X, xn → x as n → ∞, then the sequence {f$\{xn\}_{n=1}^{\infty}$ converges to f(x) in Y.

Proof. Assume that f : X → Y is continuous at a point x ∈ X. Also assume that {xn} ∞ n=1 is a sequence in X such that xn → x as n → ∞. To prove: f(xn) → f(x) in Y . So start with an open set V in Y containing f(x). Since f is continuous at x, U = f −1 (V) is an open set in X. Now f(x) ∈ V implies x ∈ f −1 (V) = U. That is U is an open set containing x. Hence xn → x implies there exists n0 ∈ N such that xn ∈ U for all n ≥ n0 . This implies f(xn) ∈ V for all n ≥ n0. That is, whenever xn → x as n → ∞ then f(xn) → f(x) as n → ∞.

Conversely, suppose that {xn} is a sequence in X, xn → x as n → ∞ implies f(xn) → f(x). Now we will have to prove that f is continuous at x. It is to be noted that to prove this converse part we will make use of the fact that X is a first countable space. Now X is a first countable space implies there exists a local base $\{Vn(x)\}_{n=1}^{\infty}$ at x such that Vn+1 ⊆ Vn for all n ∈ N. We will use the method of proof by contradiction. If f is not continuous at x then there should exist an open set W containing f(x) such that f(U) * W for any open set U containing x. In particular for such an open set W, f(Vn) * W for all n = 1, 2, 3, ... Hence there exists xn ∈ Vn such that f(xn) ∉W

Claim: xn → x as → ∞. So start with an open set V in X containing x. Now {Vn} ∞ n=1 is a local base at x implies there exists n0 ∈ N such that Vn0 ⊆ V . Hence xn ∈ Vn ⊆ Vn0 ⊆ V for all n ≥ n0. That is for each open set V containing x there exists n0 ∈ N such that xn ∈ V for all n ≥ n0. Hence xn → x as n → ∞. But this sequence {xn} in X is such that f(xn) ∈/ W, where W is an open set containing f(x). So we have arrived at a contradiction to our assumption namely xn ∈ X, xn → x ∈ X implies f(xn) → f(x). We arrived at this contradiction by assuming f is not continuous at x. Therefore our assumption is wrong and hence f is continuous at x.

Example 5.2.4. Let Jc = {A ⊆ R : Ac is countable or Ac = R}, the co-countable topology on R, and X = (R,Jc), Y = (R,Js), where Js is the standard topology on R. Define f : X → Y as f(x) = x for all x ∈ X. Suppose {xn} is a sequence in X such that {xn} converges to x ∈ X = R. Then it is easy to prove that there exists n0 ∈ N such that xn = x for all n ≥ n0. (If this statement is not true then there exists a subsequence $\{xn\}_{k=1}^{\infty}$ of $\{xn\}_{n=1}^{\infty}$ such that xnk 6= x for all k ∈ N. Then

U = RK{xnk : k ∈ N} is an open set in X containing x. Hence {xn} converges to x in X implies there exists n0 ∈ N such that xn ∈ U for all n ≥ n0. In particular for k ≥ n0, nk ≥ k ≥ n0 and this implies xnk ∈ U.) So we have the following: xn → x in X implies f(xn) → f(x) in Y. But the given function f : X → Y is not a continuous function (note: f −1 (0, 1) = (0, 1) is not an open set in (R,Jc)). This example does not give any contradiction to theorem 5.2.3. From this example we conclude that X = (R,Jc) is not a first countable topological space

8.3 REGULAR AND NORMAL TOPOLOGICAL SPACES

Definition. A topological space (X, J) is called a T1 space if for each x ∈ X, the singleton set {x} is a closed set in (X, J).

Definition A T1-topological space (X, J) is called a regular space if for each x ∈ X and for each closed subset A of X with x /∈ A, there exist open sets U, V in X satisfying the following:

(i) x ∈ U, A ⊆ V , (ii) U ∩ V = φ.

Result 5.3.3. Every regular topological space (X, J) is a Hausdorff space.

Proof. Let x, y ∈ X, x 6= y. By definition every regular space is a T1-space. Hence {y} is a closed set. Also x 6= y implies x /∈ A = {y}. Now {y} is a closed set which does not contain x. Since (X, J) is a regular space, there exist open sets U, V in X satisfying the following:

(i) x ∈ U, A = {y} ⊆ V ,
(ii) U ∩V = φ that is U, V are open sets in X such that x ∈ U, y ∈ V and U ∩V = φ. Hence (X, J) is a Hausdorff topological space.

Notes

Exercise 5.3.4. Prove that every Hausdorff space is a T1-space

Example 5.3.5. Let X be an infinite set and Jf be the cofinite topology on X. Then (X, Jf) is a T1-space, but (X, Jf) is not a Hausdorff space. For each x ∈ X, U = XK{x} is an open set. Hence U c = XKU = {x} is a closed set in X. That is for each x ∈ X, the singleton set {x} is a closed set. Therefore (X, J) is a T1 - space. Take any x, y ∈ X, x 6= y. Suppose there exist open sets U, V in X such that x ∈ U, y ∈ V and U ∩ V = φ. Now U, V are nonempty open subsets of the cofinite topological space (X, Jf) implies U c , V c are finite sets. Hence X = φ c = (U ∩ V) c = U c ∪ V c is a finite set. Therefore there cannot exist any open sets U, V in (X, Jf) satisfying x ∈ U, y ∈ V and U ∩V = φ. This means (X, Jf) is not a Hausdorff space.

Now let us give an example of a topological space which is Hausdorff but not regular. Take X = R and BK = {(a, b), (a, b)KK : a, b ∈ R, a < b}, where K = {1, 1 2 , 1 3 , . . .}. Now it is easy to prove that (left as an exercise) BK is a basis for a topology on R. Let JK be the topology on R generated by BK. If J is the usual topology on R then we know that J is generated by B = {(a, b) : a, b ∈ R, a < b}. Since we have B ⊆ BK and this implies that J = JB ⊆ JBK = JK. From this, it is clear that (R, JK) is a Hausdorff space. For x, y ∈ R, x 6= y, (R,J) is a Hausdorff space implies there exist open sets U and V in (R, J) such that x ∈ U, y ∈ V and U ∩ V = φ. But J ⊆ JK. Hence U, V ∈ JK are such that x ∈ U, y ∈ V and U ∩ V = φ and this shows that (X, JK) is a Hausdorff topological space.

Is K = {1, 1 2 , 1 3 , . . .} a closed set? Here K is a subset of R and J , JK are two different topologies on R, 0 ∈ K and 0 ∈/ K with respect to (R,J). Hence K is not a closed set in (R, J). But RKK = ∞ ∪ n=1 An, where An = (−n, n)KK for each n ∈ N. Each An is an open set in (R, JK) implies RKK is an open set in (R, JK). This implies K is a closed set in (R, JK). Also 0 ∈/ K. What are the open sets containing K? If V is an open set containing K, then for each n ∈ N, 1 n ∈ V, there exists a basic open set say (an, bn) such that 1 n ∈ (an, bn) ⊆ V (1 n ∈/ (an, bn)KK) and 0 < an < bn implies K ⊆ ∞ ∪ k=1 (ak, bk) ⊆ V.

Suppose U, V are open sets such that 0 ∈ U and K ⊆ V . Since 0 ∈ U, there exists a basic open set B such that 0 ∈ B ⊆ U. If B is of the form (a, b) then (a, b) ∩ K 6= φ. So U ∩V 6= φ. If B is of the form (a, b)KK, choose n0 ∈ N such that 1 n0 < b. Since 1 n0 ∈ V, there exists an open interval (c, d) such that 1 n0 ∈ (c, d) ⊆ V. Now since (a, b) ∩ (c, d) is not empty (it contains 1 n0), it is an interval and hence uncountable. As K is countable, ((a, b) ∩ (c, d))KK 6= φ, i.e, ((a, b)KK) ∩ (c, d) 6= φ. Therefore U ∩ V 6= φ.

So we have proved that there cannot exist open sets U, V in (R, JK) with 0 ∈ U, K ⊆ V and U ∩ V = φ. This shows that (R, JK) is not a regular space.

Check In Progress

Q. 1 Define Regular Space.

Solution :
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Q. 2 Define Countability.

Solution :
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8.4 SEPARATION AXIOM

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms. These are sometimes called *Tychonoff separation axioms*, after Andrey Tychonoff.

The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of *kinds* of topological spaces. However, the term "separation axiom" has stuck. The separation axioms are denoted with the letter "T" after the German *Trennungsaxiom*, which means "separation axiom."

The precise meanings of the terms associated with the separation axioms has varied over time, as explained in History of the separation axioms. It is important to understand the authors' definition of each condition mentioned to know exactly what they mean, especially when reading older literature.

Preliminary Definitions

Before we define the separation axioms themselves, we give concrete meaning to the concept of separated sets (and points) in topological spaces. (Separated sets are not the same as *separated spaces*, defined in the next section.)

The separation axioms are about the use of topological means to distinguish disjoint sets and distinct points. It's not enough for elements of a topological space to be distinct (that is, unequal); we may want them to be *topologically distinguishable*. Similarly, it's not enough for subsets of a topological space to be disjoint; we may want them to be *separated* (in any of various ways). The separation axioms all say, in one way or another, that points or sets that are distinguishable or separated in some weak sense must also be distinguishable or separated in some stronger sense.

Let X be a topological space. Then two points x and y in X are topologically distinguishable if they do not have exactly the same neighbourhoods (or equivalently the same open neighbourhoods); that is, at least one of them has a neighbourhood that is not a neighbourhood of the other (or equivalently there is an open set that one point belongs to but the other point does not).

Two points *x* and *y* are separated if each of them has a neighbourhood that is not a neighbourhood of the other; that is, neither belongs to the other's closure. More generally, two subsets *A* and *B* of *X* are separated if each is disjoint from the other's closure. (The closures themselves do not have to be disjoint.) All of the remaining conditions for separation of sets may also be applied to points (or to a point and a set) by using singleton sets. Points *x* and *y* will be considered separated, by neighbourhoods, by closed neighbourhoods, by a continuous function, precisely by a function, if and only if their singleton sets $\{x\}$ and $\{y\}$ are separated according to the corresponding criterion.

Subsets *A* and *B* are separated by neighbourhoods if they have disjoint neighbourhoods. They are separated by closed neighbourhoods if they have disjoint closed neighbourhoods. They are separated by a continuous function if there exists a continuous function *f* from the space *X* to the real line R such that the image $f(A)$ equals $\{0\}$ and $f(B)$ equals $\{1\}$. Finally, they are precisely separated by a continuous function if there exists a continuous function *f* from *X* to R such that the preimage $f^{-1}(\{0\})$ equals *A* and $f^{-1}(\{1\})$ equals *B*.

These conditions are given in order of increasing strength: Any two topologically distinguishable points must be distinct, and any two separated points must be topologically distinguishable. Any two separated sets must be disjoint, any two sets separated by neighbourhoods must be separated, and so on.

For more on these conditions (including their use outside the separation axioms), see the articles Separated sets and Topological distinguishability.

Main definitions

These definitions all use essentially the preliminary definitions above.

Notes

Many of these names have alternative meanings in some of mathematical literature, as explained on History of the separation axioms; for example, the meanings of "normal" and "T_4" are sometimes interchanged, similarly "regular" and "T_3", etc. Many of the concepts also have several names; however, the one listed first is always least likely to be ambiguous.

Most of these axioms have alternative definitions with the same meaning; the definitions given here fall into a consistent pattern that relates the various notions of separation defined in the previous section. Other possible definitions can be found in the individual articles.

In all of the following definitions, *X* is again a topological space.

- *X* is T_0, or *Kolmogorov*, if any two distinct points in *X* are topologically distinguishable. (It will be a common theme among the separation axioms to have one version of an axiom that requires T_0 and one version that doesn't.)
- *X* is R_0, or *symmetric*, if any two topologically distinguishable points in *X* are separated.
- *X* is T_1, or *accessible* or *Fréchet* or *Tikhonov*, if any two distinct points in *X* are separated. Thus, *X* is T_1 if and only if it is both T_0 and R_0. (Although you may say such things as "T_1 space", "Fréchet topology", and "suppose that the topological space *X* is Fréchet"; avoid saying "Fréchet space" in this context, since there is another entirely different notion of Fréchet space in functional analysis.)
- *X* is R_1, or *preregular*, if any two topologically distinguishable points in *X* are separated by neighbourhoods. Every R_1 space is also R_0.
- *X* is Hausdorff, or *T_2* or *separated*, if any two distinct points in *X* are separated by neighbourhoods. Thus, *X* is Hausdorff if and only if it is both T_0 and R_1. Every Hausdorff space is also T_1.
- *X* is $T_{2½}$, or *Urysohn*, if any two distinct points in *X* are separated by closed neighbourhoods. Every $T_{2½}$ space is also Hausdorff.
- *X* is completely Hausdorff, or *completely T_2*, if any two distinct points in *X* are separated by a continuous function. Every completely Hausdorff space is also $T_{2½}$.
- *X* is regular if, given any point *x* and closed set *F* in *X* such that *x* does not belong to *F*, they are separated by neighbourhoods.

(In fact, in a regular space, any such x and F will also be separated by closed neighbourhoods.) Every regular space is also R_1.

- X is regular Hausdorff, or T_3, if it is both T_0 and regular.[1] Every regular Hausdorff space is also $T_{2½}$.
- X is completely regular if, given any point x and closed set F in X such that x does not belong to F, they are separated by a continuous function. Every completely regular space is also regular.
- X is Tychonoff, or $T_{3½}$, *completely* T_3, or *completely regular Hausdorff*, if it is both T_0 and completely regular.[2] Every Tychonoff space is both regular Hausdorff and completely Hausdorff.
- X is normal if any two disjoint closed subsets of X are separated by neighbourhoods. (In fact, a space is normal if and only if any two disjoint closed sets can be separated by a continuous function; this is Urysohn's lemma.)
- X is normal Hausdorff, or T_4, if it is both T_1 and normal. Every normal Hausdorff space is both Tychonoff and normal regular.
- X is completely normal if any two separated sets are separated by neighbourhoods. Every completely normal space is also normal.
- X is completely normal Hausdorff, or T_5 or *completely* T_4, if it is both completely normal and T_1. Every completely normal Hausdorff space is also normal Hausdorff.
- X is perfectly normal if any two disjoint closed sets are precisely separated by a continuous function. Every perfectly normal space is also completely normal.
- X is perfectly normal Hausdorff, or T_6 or *perfectly* T_4, if it is both perfectly normal and T_1. Every perfectly normal Hausdorff space is also completely normal Hausdorff.

8.4.1 Other Separation Axioms

There are some other conditions on topological spaces that are sometimes classified with the separation axioms, but these don't fit in with the usual separation axioms as completely. Other than their definitions, they aren't discussed here; see their individual articles.

- X is sober if, for every closed set C that is not the (possibly nondisjoint) union of two smaller closed sets, there is a unique point *p* such that the closure of {*p*} equals *C*. More briefly, every irreducible closed set has a unique generic point. Any Hausdorff space must be sober, and any sober space must be T_0.

- *X* is weak Hausdorff if, for every continuous map *f* to *X* from a compact Hausdorff space, the image of *f* is closed in *X*. Any Hausdorff space must be weak Hausdorff, and any weak Hausdorff space must be T_1.
- *X* is semiregular if the regular open sets form a base for the open sets of *X*. Any regular space must also be semiregular.
- *X* is quasi-regular if for any nonempty open set *G*, there is a nonempty open set *H* such that the closure of *H* is contained in *G*.
- *X* is fully normal if every open cover has an open star refinement. *X* is fully T_4, or fully normal Hausdorff, if it is both T_1 and fully normal. Every fully normal space is normal and every fully T_4 space is T_4. Moreover, one can show that every fully T_4 space is paracompact. In fact, fully normal spaces actually have more to do with paracompactness than with the usual separation axioms.

8.5 FIRST AND SECOND COUNTABLE TOPOLOGICAL SPACES

Definition 5.1.1. A topological space (X, J) is said to have a countable local basis (or countable basis) at a point x ∈ X if there exists a countable collection say Bx of open sets containing x such that for each open set U containing x there exists V ∈ Bx with V ⊆ U.

Definition 5.1.2. A topological space (X, J) is said to be first countable or said to satisfy the first countability axiom if for each x ∈ X there exists a countable local base at x.

Examples 5.1.3. (i). Let (X, d) be a metric space then for each x ∈ X, Bx = {B(x, 1 n) : n ∈ N} is a countable local basis at x. Hence (X, Jd) is a first countable space. So, we say that every metric space (X, d) is a first countable space.

(ii). Let X = N and J = {φ, X, {1}, {1, 2}, . . . , {1, 2, . . . , n}, . . . , } then obviously (X,J) is a first countable topological space. Note that this is not an interesting example of a first countable topological space. Once the topology J is a countable collection then (X, J) is a first countable space.

Example 5.1.4. Let X = R and Jl be the lower limit topology on R generated by {[a, b) : a, b ∈ R, a < b}. For each x ∈ X, Bx = {[x, x + 1 n) : n ∈ N} is a countable 111 local base at x. Hence (R, Jl) = Rl is a first countable topological space. Now let us see a stronger version of first countable topological space.

8.5.1 Invariant properties

Unlike curves and surfaces, higher dimensional manifolds cannot be understood by means of visual intuition. Indeed, it is difficult or even impossible to decide whether two different descriptions of a higher-dimensional manifold refer to the same object. For this reason it is useful to develop concepts and criteria that describe intrinsic geometric and topological aspects of these mathematical objects. Such criteria are commonly referred to as being **invariant**, because they are the same relative to all possible descriptions of a particular manifold. Thus, it is possible to distinguish two manifolds if they disagree with respect to some invariant property. Naively, one could hope to develop an arsenal of invariant criteria that would definitively classify all manifolds up to isomorphism. Unfortunately, it is known that for manifolds of dimension 4 and higher, no single decision procedure can be used to decide whether two manifolds have the same topological configuration.

Some invariant properties are local, and serve to characterize manifolds at the smallest of scales. Other invariant properties are global, and take account of a manifold's overall spatial structure. Many invariant properties relevant to manifold theory come from point set topology. Separability of points, or the Hausdorff property is one such invariant, dimension (see below) is another. Compactness, connectedness, and paracompactness are important global properties. However, many mathematicians consider separability of points and paracompactness to be so essential that they include them in the very definition of *manifold.*

Algebraic topology is a source of a number of important global invariant properties. Some key criteria include the *simply connected* property and orientability (see below). Indeed several branches of mathematics, such as homology and homotopy theory, and the theory of characteristic classes were founded in order to study invariant properties of manifolds.

It is often said that, aside from dimension, a differential manifold has no local invariants. However, if a manifold is endowed with some geometric information, such Riemannian structure, then invariant local properties may arise. Examples include the notions of flatness and constant curvature for Riemannian manifolds, and the absence of torsion for manifolds equipped with an affine connection.

8.5.2 Orientability

In dimensions two and higher, a simple but important invariant criterion is the question of whether a manifold admits ameaningful orientation. Consider a topological manifold with charts mapping to $\mathbf{R}^n$. Given an ordered basis for $\mathbf{R}^n$, a chart causes its piece of the manifold to itself acquire a sense of ordering, which in3 dimensions can be viewed as either right-handed or left-handed. Overlapping charts are not required to agree in their sense of ordering, which gives manifolds an important freedom. For some manifolds, like the sphere, charts can be chosen so that overlapping regions agree on their "handedness"; these are *orientable* manifolds. For others, this is impossible. The latter possibility is easy to overlook, because any closed surface embedded (without self-intersection) in three-dimensional space is orientable.

Some illustrative examples of non-orientable manifolds include: (1) the Möbius strip, which is a manifold with boundary, (2) the Klein bottle, which must intersect itself in 3-space, and (3) the real projective plane, which arises naturally in geometry.

Check In Progress

Q. 1 Define Seperation Axioms.

Solution :

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Q. 2 Write Seperation Properties.

Solution :

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8.5.3 Möbius Strip

Begin with an infinite circular cylinder standing vertically, a manifold without boundary. Slice across it high and low to produce two circular boundaries, and the cylindrical strip between them. This is an orientable manifold with boundary, upon which "surgery" will be performed. Slice the strip open, so that it could unroll to become a rectangle, but keep a grasp on the cut ends. Twist one end 180°, making the inner surface face out, and glue the ends back together seamlessly. This results in a strip with a permanent half-twist: the Möbius strip. Its boundary is no longer a pair of circles, but (topologically) a single circle; and what was once its "inside" hasmerged with its "outside", so that it now has only a *single* side.

Klein bottle

Take two Möbius strips; each has a single loop as a boundary. Straighten out those loops into circles, and let the strips distort into cross-caps. Gluing the circles together will produce a new, closed manifold without boundary, the Klein bottle. Closing the surface does nothing to improve the lack of orientability, it merely removes the boundary. Thus, the Klein bottle is a closed surface with no distinction between inside and outside. Note that in three-dimensional space, a Klein bottle's surface must pass through itself. Building a Klein bottle which is not self-intersecting requires four or more dimensions of space.

Real projective plane

Begin with a sphere centered on the origin. Every line through the origin pierces the sphere in two opposite points called *antipodes*. Although there is no way to do so physically, it is possible to mathematically merge each antipode pair into a single point. The closed surface so produced is the real projective plane, yet another non-orientable surface. It has a number of equivalent descriptions and constructions, but this route explains its name: all the points on any given line through the origin project to the same "point" on this "plane".

8.5.4.1 Genus and the Euler characteristic

For two dimensional manifolds a key invariant property is the genus, or the "number of handles" present in a surface. A torus is a sphere with one handle, a double torus is a sphere with two handles, and so on. Indeed it is possible to fully characterize compact, two-dimensional manifolds on the basis of genus and orientability. In higher-dimensional manifolds genus is replaced by the notion of Euler characteristic.

8.5.4.2 Dimension

Dimensionality is built right into the definition of an *n*-manifold. Dimension is a local invariant, but it does not change as one moves inside the manifold. However in some settings it is convenient to allow a single manifold to consist of several disconnected pieces, each of its own dimension.

8.5.5 Generalizations of Manifolds

Orbifolds: An orbifold is a generalization of manifold allowing for certain kinds of " singularities" in the topology. Roughly speaking, it is a space which locally looks like the quotients of some simple space (*e.g.* Euclidean space) by the actions of various finite groups. The singularities correspond to fixed points of the group actions, and the actions must be compatible in a certain sense.

Algebraic varieties and schemes: An algebraic variety is glued together from affine algebraic varieties, which are zero sets of polynomials over algebraically closed fields. Schemes are likewise glued together from affine schemes, which are a generalization of algebraic varieties. Both are related to manifolds, but are constructed using sheaves instead of atlases. Because of singular points one cannot assume a variety is a manifold (even though linguistically the French *variété*, German *Mannigfaltigkeit* and English *manifold* are much the same thing).

CW-complexes: A CW complex is a topological space formed by gluing objects of different dimensionality together; for this reason they generally are not manifolds. However, they are of central interest in algebraic topology, especially in homotopy theory, where such dimenisonal defects are acceptable.

8.6 SUMMARY

We study in this unit about Seperation Properties and Seperatipon Axioms. We study Manifolds and Its properties. We study Generalizations of Manifolds. We Study Countability and Its properties.

8.7 KEYWORD

ORBIFOLD: An orbifold is something with many folds; unfortunately

COUNTABILITY : He fact of being countable | Meaning, pronunciation, translations and examples.

GENUS : A class of things that have common characteristics and that can be divided into subordinate kinds

8.8 QUESTIONS FOR REVIEW

1. Every regular topological space (X, J) is a Hausdorff space.
2. Let X and Y be topological spaces and further suppose X is a first countable topological space. Then a function f : X → Y is continuous at a point x ∈ X if and only if for every sequence {xn} ∞ n=1 in X, xn → x as n → ∞, then the sequence {f(xn)} ∞ n=1 converges to f(x) in Y
3. Let (X, J) be a first countable topological space and A be a nonempty subset of X. Then for each x ∈ X, x ∈ A if and only if there exists a sequence {xn} ∞ n=1 in A such that xn → x as n → ∞.

4. If (X, J) is a first countable topological space then for each $x \in X$ there exists a countable local base say $\{V_n(x)\}_{n=1}^{\infty}$ such that $V_{n+1}(x) \subseteq V_n(x)$.

8.9 SUGGESTION READING AND REFERENCES

- *Schechter, Eric (1997).* Handbook of Analysis and its Foundations. *San Diego: Academic Press.* ISBN 0126227608. (has R_i axioms, among others)
- *Willard, Stephen (1970).* General topology. *Reading, Mass.: Addison-Wesley Pub. Co.* ISBN 0-486-43479-6. (has all of the non-R_i axioms mentioned in the Main Definitions, with these definitions)
- *Merrifield, Richard E.;* Simmons, Howard E. *(1989).* Topological Methods in Chemistry. *New York: Wiley.* ISBN 0-471-83817-9. (gives a readable introduction to the separation axioms with an emphasis on finite spaces)

8.10 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 3

Q 2 Check in Section 1

Check in Progress-II

Answer Q. 1 Check in Section 4.1

Q 2 Check in Section 5.1

UNIT 9 :COUNTABILITY AND SEPARATION AXIOMS-II

SRTUCTURE

9.0 OBJECTIVES

Here In this topic we study There are two basic themes to the next several sections:

a. What properties of a topology allow us to conclude that the topology is given by a metric?

b. What properties of a space allow us to conclude that the space actually is (homeomorphic to) a subspace of R n (or at least a subspace of Rω)?

Learn Urysohn Metrization Theorem

9.1 INTRODUCTION

The topics of this chapter are less motivated by analysis and more motivated by the study of topology itself. Munkres declares “our basic goal” in this chapter is the proof of the Urysohn Metrization Theorem which deals with the condition under which a topological space can be embedded in a metric space. Another embedding theorem states that a compact m-manifold can be embedded in R N for some B ∈ N. An m-manifold is like an m-dimensional surface (manifolds are studied in detail in the area of differential geometry).

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the **separation axioms**. These are sometimes called *Tychonoff separation axioms*, after Andrey Tychonoff.

The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of *kinds* of topological spaces. However, the term "separation axiom" has stuck. The separation axioms are denoted with the letter "T" after the German *Trennungsaxiom*, which means "separation axiom."

The precise meanings of the terms associated with the separation axioms has varied over time, as explained in History of the separation axioms. It is important to understand the authors' definition of each condition mentioned to know exactly what they mean, especially when reading older literature.

9.1.1 Regular and Normal Topological Spaces

Definition.1.1 A topological space (X, J) is called a T_1 space if for each x ∈ X, the singleton set {x} is a closed set in (X, J).

Definition. 1.2 A T_1-topological space (X, J) is called a regular space if for each x ∈ X and for each closed subset A of X with x /∈ A, there exist open sets U, V in X satisfying the following:

(i) x ∈ U, A ⊆ V , (ii) U ∩ V = φ.

Result 1.3. Every regular topological space (X, J) is a Hausdorff space.

Proof. Let x, y ∈ X, x 6= y. By definition every regular space is a T_1-space. Hence {y} is a closed set. Also x 6= y implies x ∉ A = {y}. Now {y} is a closed set which does not contain x. Since (X, J) is a regular space, there exist open sets U, V in X satisfying the following:

(i) x ∈ U, A = {y} ⊆ V ,
(ii) U ∩V = φ that is U, V are open sets in X such that x ∈ U, y ∈ V and U ∩V = φ. Hence (X, J) is a Hausdorff topological space.

Exercise 1.4. Prove that every Hausdorff space is a T1-space.

Example 1.5. Let X be an infinite set and Jf be the cofinite topology on X. Then (X, Jf) is a T_1-space, but (X, Jf) is not a Hausdorff space. For each x ∈ X, U = XK{x} is an open set. Hence U c = XKU = {x} is a closed set in X. That is for each x ∈ X, the singleton set {x} is a closed set. Therefore (X, J) is a T_1 - space. Take any x, y ∈ X, x 6= y. Suppose there exist open sets U, V in X such that x ∈ U, y ∈ V and U ∩ V = φ. Now U, V are nonempty open subsets of the cofinite topological space (X, Jf) implies U c , V c are finite sets. Hence X = φ c = (U ∩ V) c = U c ∪ V c is a finite set. Therefore there cannot exist any open sets U, V in (X, Jf) satisfying x ∈ U, y ∈ V and U ∩V = φ. This means (X, Jf) is not a Hausdorff space.

Now let us give an example of a topological space which is Hausdorff but not regular. Take X = R and BK = {(a, b), (a, b)KK : a, b ∈ R, a < b}, where K = {1, 1 2 , 1 3 , . . .}. Now it is easy to prove that (left as an exercise) BK is a basis for a topology on R. Let JK be the topology on R generated by BK. If J is the usual topology on R then we know that J is generated by $B = \{(a, b) : a, b \in R, a < b\}$. Since we have $B \subseteq BK$ and this implies that $J = JB \subseteq JBK = JK$.

From this, it is clear that (R, JK) is a Hausdorff space. For $x, y \in R$, $x 6= y$, (R,J) is a Hausdorff space implies there exist open sets U and V in (R, J) such that $x \in U$, $y \in V$ and $U \cap V = \varphi$. But $J \subseteq JK$. Hence $U, V \in JK$ are such that $x \in U$, $y \in V$ and $U \cap V = \varphi$ and this shows that (X, JK) is a Hausdorff topological space.

Is K = {1, 1 2 , 1 3 , . . .} a closed set? Here K is a subset of R and J , JK are two different topologies on R, $0 \in K$ and $0 \notin K$ with respect to (R,J). Hence K is not a closed set in (R, J). But RKK = $\bigcup_{n=1}^{\infty}$. An, where An = (−n, n)KK for each $n \in N$. Each An is an open set in (R, JK) implies RKK is an open set in (R, JK). This implies K is a closed set in (R, JK). Also $0 \notin K$. What are the open sets containing K? If V is an open set containing K, then for each $n \in N$, $1\ n \in V$, there exists a basic open set say (an, bn) such that $1\ n \in (an, bn) \subseteq V$ (1 n ∈/ (an, bn)KK) and 0 < an < bn implies $K \subseteq \bigcup_{k=1}^{\infty} (ak, bk) \subseteq V$.

Suppose U, V are open sets such that $0 \in U$ and $K \subseteq V$. Since $0 \in U$, there exists a basic open set B such that $0 \in B \subseteq U$. If B is of the form (a, b) then $(a, b) \cap K$ 6= φ. So $U \cap V$ 6= φ. If B is of the form (a, b)KK, choose $n0 \in N$ such that 1 n0 < b. Since $1\ n0 \in V$, there exists an open interval (c, d) such that $1\ n0 \in (c, d) \subseteq V$. Now since $(a, b) \cap (c, d)$ is not empty (it contains 1 n0), it is an interval and hence uncountable. As K is countable, $((a, b) \cap (c, d))$KK 6= φ, i.e, ((a, b)KK) $\cap$ (c, d) 6= φ. Therefore $U \cap V$ 6= φ.

So we have proved that there cannot exist open sets U, V in (R, JK) with $0 \in U$, $K \subseteq V$ and $U \cap V = \varphi$. This shows that (R, JK) is not a regular space.

Definition 1.6. A topological space (X,J) is said to be a normal space if and only if it satisfies:

(i) (X,J) is a T_1-space,

(ii) A, B closed sets in X, A ∩ B = φ implies there exist open sets U, V in X such that A ⊆ U, B ⊆ V and U ∩ V = φ.

Remark 1.7. It is to be noted that every normal space is a regular space.

Theorem 1.8. Every metric space (X, d) is a normal space, That is if Jd is the topology induced by the metric then the topological space (X, Jd) is a normal space.

Proof.

Let A, B be disjoint closed subsets of X. Then for each a ∈ A, a ∉ B = B implies d(a, B) = inf{d(a, b) : b ∈ B} > 0. If ra = d(a, B) > 0 then B(a, ra) ∩ B = φ (if there exists b0 ∈ B such that d(b0, a) < ra, then ra = d(a, B) ≤ d(a, b0) < ra a contradiction). Similarly for each b ∈ B there exists rb > 0 such that B(b, rb)∩A = φ. Let $U = \bigcup_{a\in A} B(a, ra\ 3)$, $V = \bigcup_{b\in B} B(a, rb\ 3)$.

Now it is easy to prove that U ∩ V = φ. Hence if A, B are disjoint closed subsets of X then there exist open sets U, V in X such that A ⊆ U, B ⊆ V and U ∩V = φ. This implies (X, Jd) is a normal space.

Theorem1.9. A T1-topological space (X,J) is regular if and only if whenever x is a point of X and U is an open set containing x then there exists an open set V containing x such that V ⊆ U

Proof. Assume that (X,J) is a regular topological space, x ∈ X and U is an open set containing x. Now x ∈ U implies x /∈ A = U c = XKU, the complement of the open set U. Now A is a closed set and x /∈ A. Hence X is a regular space implies there exist open sets V and W of X such that x ∈ V , A = U c ⊆ W and V ∩ W = φ. Now V ∩ W = φ implies V ⊆ Wc ⊆ U (we have U c ⊆ W), V ⊆ Wc implies V ⊆ Wc = Wc (W is an open set implies Wc is a closed set) implies V ⊆ U. Hence for x ∈ X and for each open set U containing x, there exists an open set V containing x such that V ⊆ U.

Notes

Now let us assume that the above statement is satisfied. Our aim here is to prove that (X,J) is a regular space. So take a closed set A of X and a point x ∈ XKA. Now A is a closed subset of X implies U = XKA is an open set containing x. Hence by our assumption there exists an open set V containing x such that V ⊆ U = Ac . Now V ⊆ Ac implies A ⊆ (V) c = XKV . So V and (V) c = W are open sets satisfying x ∈ V, A ⊆ W and V ∩ W = V ∩ (V) c ⊆ V ∩ V c = φ. (V ⊆ V implies (V) c ⊆ V c .) Hence by definition (X,J) is a regular space.

In a similar way we prove the following theorem.

Theorem 1.10. A T1-topological space is a normal space if and only if whenever A is a closed subset of X and U is an open set containing A, then there exists an open set V containing A such that V ⊆ U.

Proof. Assume that (X,J) is a normal topological space. Now take a closed set A and an open set U in X such that A ⊆ U. Now A ⊆ U implies U c ⊆ Ac . Here A, Uc = B are closed sets such that A ∩ B = A ∩ U c ⊆ U ∩ U c = φ. That is A, B are disjoint closed subsets of the normal space (X,J). Hence there exist open sets U, W in X such that A ⊆ V, B = U c ⊆ W and V ∩W = φ. Further V ⊆ Wc (note: V ⊆ Wc implies V ⊆ Wc = Wc). Now V ⊆ Wc ⊆ U. Hence whenever A is a closed set and U is an open set containing A then there exists an open set V such that A ⊆ V, V ⊆ U. Now let us assume that the above statement is satisfied. So our aim is to prove that (X,J) is a normal space. So start with disjoint closed subsets say A, B of X. Now A ∩ B = φ implies A ⊆ Bc = U. That is U is an open set containing the closed set A. Hence by our assumption there exists an open set V such that A ⊆ V, V ⊆ U. Now V ⊆ U implies U c ⊆ (V) c implies B ⊆ (V) c . Further V ∩ (V) c ⊆ V ∩ V c = φ. That is whenever A, B are closed subsets of X, then there exist open sets V and (V) c = W such that A ⊆ V, B ⊆ W and V ∩ W = φ. Therefore by definition (X,J) is a normal space.

9.2 EXAMPLE OF A TOPOLOGICAL SPACE WHICH IS REGULAR BUT NOT NORMAL

Let Jl be a lower limit topology on R. That is Rl = (R,Jl). Now let us prove that the product space Rl × Rl is a regular space. (If X, Y are regular topological spaces then the product space X × Y is a regular space. Hence it is enough to prove that Rl is a regular space.) For (x, y) ∈ R 2 , each basic open set U of the form U = [x, a) × [y, b) is both open and closed. Hence for each basic neighbourhood U of (x, y) in Rl × Rl there exists a neighbourhood V = U of (x, y) such that V = U ⊆ U. Now if U 0 is any open set containing (x, y) then there exists a basic open set U as given above such that (x, y) ∈ U = [x, a) × [y, b) ⊆ U 0 . Therefore V = U is an open set containing (x, y) and V = U = U ⊆ U 0 . Also Rl × Rl is a Hausdorff space. Hence Rl × Rl is a regular space. Now let us take Y = {(x, y) ∈ R 2 : y = −x} then for each (x, y) ∈ Y there exists a, b ∈ R, x < a, y < b such that ([x, a) × [y, b))∩Y = {(x, y)}. Hence each singleton {(x, y)} is open in the subspace Y of Rl × Rl . This proves that the subspace Y of Rl × Rl is discrete. Also Y is a closed subset of Rl × Rl . Let A = {(x, y) ∈ R 2 : y = −x ∈ Q}, B = {(x, y) ∈ R 2 : y = −x ∈ Qc}. Now A, B are closed sets in Y and Y is a closed set in Rl×Rl implies A, B are closed in Rl×Rl . Also A ∩ B = φ. Suppose there exist open sets U, V in Rl × Rl satisfying A ⊆ U, B ⊆ V. Then we can observe that U ∩ V 6= φ. Therefore the product space Rl × Rl is not a normal space.

Remark 2.1. We can prove that (R,Jl) = Rl is a normal space. So, Rl × Rl is a regular space but it is not a normal space.

We have already proved that every compact subset of Hausdorff topological space is closed. Essentially we use the same proof technique used there to prove the following theorem:

Theorem 2.2. Every compact Hausdorff topological space (X,J) is a regular space.

Proof. Let A be a closed subset of X and x ∈ XKA, then for each y ∈ A, x 6= y. Hence X is a Hausdorff space implies that there exist open

Notes

sets Uy, Vy in X satisfying x ∈ Uy, y ∈ Vy and Uy ∩ Vy = φ. We know that closed subset of a compact space is compact. Here A ⊆ ∪ y∈A Vy. That is {Vy : y ∈ A} is an open cover for the compact space A. Therefore there exists n ∈ N and y1, y2, . . . , yn ∈ A such that A ⊆ $\bigcup_{i=1}^{n} V_{y_i}$. Let U = $\bigcap_{i=1}^{n} U_{y_i}$ and V = $\bigcup_{i=1}^{n} V_{y_i}$. Then U, V are open sets in X satisfying x ∈ U, A ⊆ V and U ∩ V ⊆ U ∩ (Vy1 ∪ Vy2 ∪ · · · ∪ Vyn) = (U ∩ Vy1) ∪ (U ∩ Vy2) ∪ · · · ∪ (U ∩ Vyn) ⊆ (Uy1 ∩ Vy1) ∪ (Uy2 ∩ Vy2) ∪ · · · ∪ (Uyn ∩ Vyn) = φ. Hence by definition (X,J) is a regular space.

Now let us prove that every compact Hausdorff space is a normal space.

Theorem 2.3. Every compact Hausdorff space (X,J) is a normal space.

Proof. Let A, B be disjoint closed sets in X. Then for each x ∈ A, x /∈ B. Now (X,J) is a regular space implies there exist open sets Ux, Vx satisfying: x ∈ Ux; B ⊆ Vx and Ux ∩ Vx = φ. Now {Ux : x ∈ A} is an open cover for A implies there exists n ∈ N, x1, x2, . . . , xn ∈ A such that A ⊆ $\bigcup_{i=1}^{n} U_{x_i}$. Let U = Ux1 ∪ Ux2 ∪ · · · ∪ Uxn and V = Vx1 ∩Vx2 ∩ · · · ∩ Vxn . Then U, V are open sets in X satisfying A ⊆ U, B ⊆ V and U ∩ V = φ. Hence by definition (X,J) is a normal space.

Theorem 2.4. Closed subspace of a normal topological space (X,J) is normal.

Proof. Let Y be a closed subspace of (X,J). That is Y is a closed subset of (X,J) and JY = {A ∩Y : A ∈ J } is a topology on Y . So we will have to prove that (Y,JY) is a normal space. To prove this, take a closed set A ⊆ Y and an open set U in (Y,JY) such that A ⊆ U. Now U is an open set in (Y,JY) implies there exists V ∈ J such that U = V ∩Y . Also A is a closed set in the subspace implies A = AY = A∩Y (here AY denotes the closure of A in (Y,JY) and A denotes the closure of A in (X,J)). Now A, Y are closed sets in X implies A ∩ Y is also a closed set in X. Hence A is a closed set in (X,J) and V is an open set in (X,J) containing A and (X,J) is a normal topological space implies there exists an open set W in (X,J) such that A ⊆ W and W ⊆ V. Now W ∩ Y is an open set in (Y,JY) and A ⊆ W ∩ Y and W ∩ Y ⊆ W ∩ Y ⊆ V ∩ Y ⊆ U. We started with a closed set A in (Y,JY) and an open set U in (Y,JY) such that A ⊆ U. Now we have proved that there exists an open set W ∩ Y in (Y,JY) satisfying A ⊆ W ∩ Y and (W ∩ Y)Y = W ∩ Y ∩ Y = W ∩ Y ⊆ U. That is W ∩ Y is an open set in the subspace containing A and closure of this open set with respect to the subspace (Y,JY) is contained in U. Hence (Y,JY) is a normal space.

Check In Progress-I

Q. 1 Every compact Hausdorff space (X,J) is a normal space.

Solution

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Q. 2 y metric space (X, d) is a normal space, That is if Jd is the topology induced by the metric then the topological space (X, Jd) is a normal space.

Solution

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9.3 URYSOHN LEMMA

One of the most important features of normal spaces is that normality is the suitable condition to prove the very useful Urysohn's Lemma.

Now let us prove the following important theorem known as Urysohn lemma.

Theorem 3.1. Let (X, J) be a normal space and A, B be disjoint nonempty closed subsets of X. Then there exists a continuous function f : X → [0, 1] such that f(x) = 0 for every x in A, and f(x) = 1 for every x in B.

Proof. A ∩ B = φ implies A ⊆ Bc = XKB. Hence Bc is an open set containing the closed set A. Now X is a normal space implies there exists an open set U0 such that A ⊆ U0 and U0 ⊆ Bc = U1. Now [0, 1] ∩ Q is a countable set implies there exists a bijective function say f : N → [0, 1] ∩ Q satisfying f(1) = 1, f(2) = 0 and f(NK{1, 2}) = (0, 1) ∩ Q. That is [0, 1] ∩ Q = {r1, r2, r3, . . .} such that r1 = 1, r2 = 0 and f(k) = rk for k ≥ 3.

Aim: To define a collection {Up}p∈[0,1]∩Q of open sets such that for p, q ∈ [0, 1]∩ Q, p < q implies Up ⊆ Uq.

Let Pn = {r1, r2, . . . , rn}. Assume that Up is defined for all p ∈ Pn, where n ≥ 2 and this collection satisfies the property namely p, q ∈ [0, 1] ∩ Q, p < q implies Up ⊆ Uq.

Note that this result is true when n = 2. Now let us prove this result for Pn+1. Here Pn+1 = Pn ∪ {rn+1}.

Let p, q ∈ Pn+1 be such that p = max{r ∈ Pn+1 : r < rn+1} and q = min{r ∈ Pn+1 : r > rn+1}. Now p, q 6= rn+1 implies p, q ∈ Pn. By our assumption Up, Uq are known and Up ⊆ Uq. Now Uq is an open set containing the closed set Up and X is a normal space. Hence there exists an open set say Urn+1 such that Up ⊆ Urn+1 and Urn+1 ⊆ Uq. If r, s ∈ Pn then we are through. Suppose r ∈ Pn and s = rn+1 then r ≤ p or r ≥ q. If r ≤ p, Ur ⊆ Up ⊆ Up ⊆ Us. If r ≥ q, Us ⊆ Uq ⊆ Uq ⊆ Ur and therefore by induction Up is defined for all p ∈ [0, 1] ∩ Q and p, q ∈ [0, 1] ∩ Q, p < q implies Up ⊆ Uq. Now define Up = φ, if p ∈ Q, p < 0 and Up = X if p ∈ Q, p > 1. Then p, q ∈ Q, p < q implies Up

$\subseteq U_q$. Define $f : X \to [0, 1]$ as $f(x) = \inf\{p \in Q : x \in U_p\}$. Now $x \in A$, then $x \in U_0$. Hence $x \in U_p$ for all $p \geq 0$. In this case $\{p \in Q : x \in U_p\} = [0,\infty) \cap Q$. Hence $\inf\{p \in Q : x \in U_p\} = 0$. That is $x \in A$ implies $f(x) = 0$. Now suppose $x \in B = U^c_1$ then $x \notin U_p$ for all $p \leq 1$. Hence $\{p \in Q : x \in U_p\} = [1,\infty) \cap Q$ implies $f(x) = 1$ for all $x \in B$.

Now let us prove that f is a continuous function. $S = \{[0, a),(a, 1] : 0 < a < 1\}$ is a subbase for [0, 1]. Hence it is enough to prove that for each a, $0 < a < 1$, $f^{-1}([0, a))$ and $f^{-1}((a, 1]))$ are open sets in X. For $0 < a < 1$, let us prove that $f^{-1}([0, a)) = \{x \in X : 0 \leq f(x) < a\} = \cup$ p< a implies there exists a rational number p such that $f(x) < p < a$. By the definition of f(x), $x \in U_p$.

Hence $f^{-1}([0, a)) \subseteq \cup$ p

Now let $x \in U_p$ for $p < a$ implies $f(x) \leq p$ implies $x \in f^{-1}([0, a))$. Hence we have

$\cup$ p< a < 1. Now $f : X \to [0, 1]$ such that inverse image of each subbasic open set is an open set implies that $f : X \to [0, 1]$ is a continuous function.

Theorem 3.2. Let (X, J) be a normal space and A, B be disjoint nonempty closed subsets of X. Then for a, b ∈ R, a < b there exists a continuous function f : X → [a, b] such that f(x) = a for every x in A, and f(x) = b for every x in B.

Proof. Define $g : [0, 1] \to [a, b]$ as $g(t) = a + (b - a)t$ then g is continuous. Now by theorem 3.1 there is a continuous function $f_1 : X \to [0, 1]$ such that $f_1(x) = 0$, for all $x \in A$ and $f_1(x) = 1$ for all $x \in B$. The function $f = g \circ f_1 : X \to [a, b]$ is a continuous function and further $f(x) = g(f_1(x)) = g(0) = a$ for all $x \in A$ and $f(x) = g(f_1(x)) = g(1) = b$ for all $x \in B$.

Remark 3.3. Let A, B be nonempty disjoint closed subsets of a metric space (X, d). Define $f : X \to R$ as $f(x) = d(x, A)\ d(x, A) + d(x, B)$. Observe that f is a continuous function satisfying the condition that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. It shows that the proof of Urysohn lemma is trivial (or say simple) if our topological space is a metrizable topological space.

9.4 TIETZE EXTENSION THEOREM

Our final application of Urysohn's lemma is an extension theorem. Extending continuous functions is a useful tool in many applications of topology. A consequence of Urysohn's lemma is the following important theorem, that says that in normal spaces, real functions on a closed subset can be always extended to the whole space.

Theorem 4.1. Tietze Extension Theorem. Let A be a nonempty closed subset of a normal space X and let f: A → [−1, 1] be a continuous function. Then there exists a continuous function g : X → [−1, 1] such that g(x) = f(x) for all x in A.

Proof. The sets −1, −1 3 , 1 3 , 1 are closed subsets of [−1, 1] and f : A → [−1, 1] is a continuous function implies A1 = f^{-1} {1 3 , 1 } B1 = f^{-1} {−1, −1 3} are closed subsets of the subspace A. (Here consider A as a subspace of X.) Now x ∈ A1 ∩ B1 implies f(x) ∈ {−1, −1 3} ∩ {1 3 , 1} a contradiction. Hence A1 ∩ B1 = φ. Now A1, B1 are closed in A and A is closed in X implies A1, B1 are closed in the normal space X. Hence by Urysohn's lemma there exists a continuous function f1 : X → {−1 3 , 1 3} such that f1(A1) = 1 3 and f1(B1) = − 1 3 then |f(x) − f1(x)| ≤ 2 3 for all x ∈A

Now consider the function f − f1 : A →{ −2 3 , 2 3} then A2 = (f − f1) {−1 2 9 , 2 3} and B2 = (f − f)$^{-1\{}$ −2 3 , −2 9} are disjoint closed subsets of X. By Urysohn lemma there exists a continuous function f2 : X → − 2 9 , 2 9 such that f2(A2) = 2 9 and f2(B2) = − 2 9 . Also |f(x) − (f1(x) + f2(x))| ≤ 4 9 for all x ∈ A. By proceeding as above, by induction, for each n ∈ N there exists a continuous function fn : X →{ h −2 n−1 3n , 2 n−1 3n i} such that

f(x) − Xn i=1 fi(x)

$\leq \frac{2}{3}^n$ for all x ∈ A. (5.4) That is fn : X → [−1, 1] is a sequence of continuous functions such that |fn(x)| ≤ 2 n−1 3n = Mn and P∞ n=1 Mn < ∞. By Weierstrass M-test, the series P∞ n=1 fn(x) converges uniformly on X. That is, if sn(x) = Pn i=1 fi(x), x ∈ X then sn(x) converges uniformly on X. Also each sn : X → R is continuous.

We know, from analysis, if a sequence sn : X → R of continuous functions converges uniformly to a function g : X → R then g is also a continuous function. Hence g : X → R be defined as g(x) = P∞ n=1 fn(x) is continuous. Now for each x ∈ A

f(x) − Xn i=1 fi(x) ≤ 2 3 n

Therefore |g(x) − f(x)| = limn→∞ Xn i=1 fi(x) − f(x) = limn→∞ Xn i=1 fi(x) − f(x) ≤ limn→∞ 2 3 n = 0. This implies g(x) = f(x) for all x ∈ A.

Definition 4.2. A topological space (X, J) is said to be completely regular if

(i) for each x ∈ X, singleton {x} is closed in (X, J) (that is (X, J) is a T1-space),

(ii) for x ∈ X and any nonempty closed set A with x /∈ A there exists a continuous function f : X → [0, 1] such that f(x) = 0 and f(y) = 1 for all y ∈ A

Result 4.3. Every normal space (X, J) is completely regular.

Proof. Let x ∈ X and A be a nonempty closed set with x /∈ A. Now {x}, A are disjoint closed sets. Hence by Urysohn's lemma there exists a continuous function f : X → [0, 1] such that f(x) = 0 and f(y) = 1 for all y ∈ A.

Result 5.5.4. If Y is a subspace of a completely regular space (X, J) then (Y , JY) is also a completely regular space.

Proof. Let y ∈ Y and A be a closed set in (Y , JY) with y /∈ A. Since A is a closed set in Y there exists a closed set F in (X, J) such that A = F ∩Y, y /∈ F, F is a closed set in the completely regular space (X, J) implies there exists a continuous function f : X → [0, 1] such that f(y) = 0 and f(a) = 1 for all a ∈ F. Now f : X → [0, 1] is a continuous function implies f|Y = g : (Y , JY) → [0, 1] (here g(x) = (f|Y)(x) = f(x) for all x ∈ Y) is a continuous function. Now g: (Y , JY) → [0, 1] is a continuous function such that g(y) = f(y) = 0 and g(a) = f(a) = 1 for all a ∈ A = F ∩ Y . Also subspace of a T1-space (do it as an exercise) is T1-space. Hence the subspace (Y , JY) is a completely regular space.

Now we are in a position to prove Urysohn metrization theorem that gives sufficient conditions under which a topological space is metrizable.

Check In Progress-II

Q. 1 **Every normal space (X, J) is completely regular**.

Solution

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Q. 2 State **Tietze Extension Theorem**.

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9.5 URYSOHN METRIZATION THEOREM

We will now see Urysohn's metrization theorem. Recall that a countable product of metric spaces is always metrizable: if X = Πn∈NXn and dn(a) = min{dn(a), 1} is the standard bounded metric associated to the metric

d_n on X_n, then we can define a metric (inducing the topology) on X for instance by

Theorem : Urysohn Metrization Theorem. Every normal space (X, J) with a countable basis is metrizable.

Proof. Let B = {B_1, B_2, . . . , } be a countable basis for (X, J). Suppose n, m ∈ N are such that $B_n \subseteq B_m$ then $B_n \cap B^c_m = \varphi$. Hence by Urysohn's lemma there exists a continuous function say $g_{n,m} : X \to R$ such that

$$g_{n,m}(x) = 0 \text{ for all } x \in B^c_m, \qquad 1$$

and

$$g_{n,m}(x) = 1 \text{ for all } x \in B_n. \qquad 2$$

Now take $x_0 \in X$ and an open set U containing x_0. Since B is a basis for (X, J) there exists $B_m \in B$ such that $x_0 \in B_m \subseteq U$. Now B_m is an open set containing x_0 implies there exists an open set V containing x_0 such that $V \subseteq B_m$. Hence there exists a basic open set B_n containing x_0 such that $B_n \subseteq V \subseteq B_m$. Hence for such pair (n, m) we have a continuous function $g_{n,m} : X \to R$ satisfying eq. 1

So if $x_0 \in X$ and U is an open set containing x_0 then there exists a continuous function $g_{n,m} : X \to R$ such that $g_{n,m}(x_0) = 1$ and $g_{n,m}(x) = 0$ for all $x \in U^c \subseteq B^c_m$. So we have proved that there exists a countable collection of continuous functions $f_n : X \to [0, 1]$ such that for $x_0 \in X$ and open set U containing x_0, there exists $n \in N$ such that $f_n(x_0) = 1 > 0$ and $f_n(x) = 0$ for all $x \in U^c$. It is to be noted that {(n, m) : n, m ∈ N} is a countable set. We know that (refer chapter 2, and exercise 9 of chapter 5) $R^w = R \times R \times R \times \cdots$ with product topology is metrizable. That is there is a metric d on R^w such that J_d, the topology on R^w induced by d, coincides with the product topology on R^w.

Now let us define a map $T : X \to R^w$ as $T(x) = (f_1(x), f_2(x), \ldots ,)$ and using this map we define $d_1(x, y) = d(T(x), T(y))$ and conclude that $J_{d_1} = J$. This will prove that (X, J) is a metrizable

Notes

topological space. Now let us prove that (X,J) is homeomorphic to a subspace of R w. Each fn : X → R is a continuous function implies T(x) = (f1(x), f2(x), . . .) is a continuous function.

To prove T is injective (one-one).

Let x, y ∈ X be such that x 6= y. Then there exist open sets U, V ∈ X such that x ∈ U, y ∈ V and U ∩ V = φ. Now U is an open set containing x implies there exists n ∈ N such that fn(x) = 1 and fn(y) = 0 (note that y ∈ U c). This implies fn(x) 6= fn(y) for this particular n ∈ N and hence (f1(x), f2(x), . . . , fn(x), . . .) 6= (f1(y), f2(y), . . . , fn(y), . . .). This means T x 6= T y. That is x, y ∈ X, x 6= y implies T x 6= T y. This implies T is 1-1.

Now it is enough to prove that T maps open set A in X to an open set T(A) in Y = T(X). Let A be an open set and y0 ∈ T(A). Now y0 ∈ T(A) implies there exists x0 ∈ A such that T(x0) = y0. Now x0 ∈ A, A is an open set implies there exists n0 ∈ N such that fn0 (x0) = 1 and fn0 (x) = 0 for all x ∈ Ac . We know that for each n ∈ N the projection map pn : R w → R defined as pn ((xk) ∞ k=1) = xn is a continuous map. Hence (0, ∞) is an open set implies V = p −1 n0 ((0,∞)) is an open subset of R w. This implies V ∩ Y is an open set in Y .

Now let us prove that y0 ∈ V ∩Y and V ∩Y ⊆ T(A). pn0 (y0) = (pn0 ·T)(x0) = fn0 (x0) = 1 > 0 implies y0 ∈ V . Also y0 ∈ Y . Hence y0 ∈ V ∩ Y . That is V ∩ Y is an open set in Y containing the point y0.

Now we claim that V ∩Y ⊆ T(A). So, let y ∈ V ∩Y . Then there exists x ∈ X such that y = T x. This implies pn0 (y) ∈ (0,∞) and pn0 (y) = pn0 (T(x)) = fn0 (x) ∈ (0,∞). Hence x ∈ A (fn0 (x) = 0 for x ∈ Ac). So we have proved that y = T x ∈ V ∩Y implies y = T x ∈ T(A). Hence V ∩ Y is an open set in Y containing T x and this set is contained in T(A). Therefore T(A) is open in Y . Hence we have proved that T : (X,J) onto —→ (Y, dY) is a homeomorphism. (Here (Y, dY) is a subspace of (R w, d).) Now d1(x, y) = d(T x, T y) for all x, y ∈ X implies d1 is a metric on X. Also it is easy to see that a subset A of X is open in (X,J) if and only if A is open in (X,Jd1). Therefore Jd1 = J .

Theorem 5.1. Let X be compact Hausdorff. Then X is normal.

Proof. We have seen that in a Hausdorff space compact (disjoint) sets are separated by open sets. Since any closed subset of X is compact, as X is compact, normality follows.

Corollary 5.2. Any locally compact Hausdorff space is regular.

Proof. It is a subspace of a normal, hence regular, space hence it is regular. (Alternatively, any locally compact Hausdorff has a basis of neighborhoods with compact closure: given x ∈ X and U a neighborhood of x, there exists open B, with x ∈ B and B ⊂ U, ie., X is regular.) A locally compact Hausdorff space needs not be normal: e.g., take]0, 1[J⊂ [0, 1]J with J uncountable. Then]0, 1[J is locally compact Hausdorff as it is open in [0, 1]J , which is compact Hausdorff (hence normal). But]0, 1[JRJ not normal.

Examples 5.3. 1. Rn is normal.

2. Any discrete space is normal - metrizable.

3. RN is normal - metrizable (any countable product of metrizable spaces is metrizable).

4. RJ with the uniform topology is normal (not with product!)

Corollary 5.4. X regular and second countable ⇔ X metrizable and separable . Proof. Recall that if X is metrizable, then second countable ⇔ separable.

Corollary 5.5. Let X be compact Hausdorff. Then X metrizable ⇔ second countable.

Example 5.6. Recall that a n-manifold M is a Hausdorff, second countable space where each point has a neighborhood diffeomorphic to an open subset of Rn . Then, since the closed ball in Rn is compact (Heine-Borel’s theorem), we can show that M is locally compact, hence regular. It follows that any n-manifold is metrizable. Another nice application of Urysohn's Lemma is a short proof that any compact n-manifold can be embedded in RN, for some N

9.6 SUMMARY

We learnt in this unit Urysohn Metrization theorem and its proof. We study Tietze Extension theorem and its proof. We study Urysohn Lemma and its proof. We study Regular and normal Topological Space.

9.7 KEYWORD

Metrization : A metrizable space is a topological space that is homeomorphic to a metric space

Urysohn : A topological space is termed a **Urysohn** space if, for any two distinct points

Hausdorff space. In topology and related branches of mathematics, a *Hausdorff space*, separated *space* or T_2 *space* is a *topological space* where for any two distinct points there exists a neighbourhood of each which is disjoint from the neighbourhood of the other

9.8 QUESTIONS FOR REVIEW

1. Discrete spaces always first countable (are metrizable): {x} is a basis at x, but second countable if, and only if, countable

2. Rn is first and second countable:]p1, q1[×...×]pn, qn[, pi , qi ∈ Q, i = 1, ..., n is a countable basis for the standard topology

3. An uncountable set with the co-finite / co-countable topology is not first countable: if B is a basis, x ∈ X, have {x} = ∩B∈BB (it is T1 - see next section), if B is countable then X \ {x} is countable union of finite/countable sets, hence countable.

4. R2 with the topology given by slotted discs: a basic neighborhhod of x ∈ R2 is given by {x} ∪ B with B an open ball with straight lines through x removed: not first countable.

5. RN the space of real sequences, with the product topology, is second countable: a basis is given by sets of the form ΠUn such that Uni =]pi , qi[, pi , qi ∈ Q, i = 1, ..., k, and Un = R, n , ni .

6. RN ρ the space of real sequences, with the uniform topology, is not second countable: {0, 1} N is an uncountable discrete subset, as ρ(x, y) = 1 for any sequences x , y, hence Bρ(x, 1) = {x} is open. It is first countable, as it is a metric space.

7. Let X be a T1 space.

(i) X is regular ⇔ x ∈ X, U nbhd of x, there exists nbhd W of x such that W ⊂ U.

(ii) X is normal ⇔ A ⊂ X closed, U ⊃ A open, there exists open W ⊃ A such that W ⊂ U.

8. Let X have a countable basis (ie, second countable). Then X regular ⇔ X normal.

9. Any locally compact Hausdorff space is regular.

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9.10 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 2

Q 2 Check in Section 2

Check in Progress-II

Answer Q. 1 Check in Section 4

Q 2 Check in Section 4

UNIT 10: SECOND COUNTABLE SPACE

SRTUCTURE

10.0 OBJECTIVE

- Learn Second Countable Space and its properties
- Learn Urysohn’s Lemma and Embedding Theorem
- Learn state and prove Metrization Theorem

Learn equivalent condition for a space to be Tychonoff

10.1 INTRODUCTION

The Urysohn Metrization Theorem tells us under which conditions a topological space X is metrizable, i.e. when there exists a metric on the underlying set of X that induces the topology of X. The main idea is to impose such conditions on X that will make it possible to embed X into a metric space Y , by homeomorphically identifying X with a subspace of Y .

Let us start with some definitions. A T1-space X (i.e. the space in which one-point sets are closed) is said to be regular if for any point x ∈ X and any closed set B ⊂ X not containing x, there exist two disjoint open sets containing x and B respectively. The space X is said to be normal if for any two disjoint closed sets B1 and B2 there exist two disjoint open sets containing B1 and B2 respectively.

10.2 URYSOHN METRIZATION THEOREM

Example. An Example of a Hausdorff space which is not normal is given by the set R, where the usual topology is enhanced by requiring that the set {1/n | n ∈ N} is closed. Examples of spaces which are regular but not normal exist, but are complicated.

Lemma. Every regular space with a countable basis is normal.

Proof. First, using regularity and countable basis, construct a countable covering {Ui} of B1 by open sets whose closures do not intersect B2. Similarly, construct an open countable covering {Vi} of B2 disjoint from B1. Then define

U 0 n := Un \ [n i=1 V¯ i and V 0 n := Vn \ [n i=1 U¯ i .

Show that these sets are open and the the collection {U 0 n} covers B1 and {V 0 n} covers B2. Finally show that U 0 := ∪U 0 n and V 0 := ∪V 0 n are disjoint. Next, we will prove one of the very deep basic results.

Urysohn lemma. Let X be a normal space, and let A and B be disjoint closed subsets of X. There exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for every $x \in A$, and $f(x) = 1$ for every $x \in B$.

Proof. Let Q be the set of rational numbers on the interval [0, 1]. For each rational number q on this interval we will define an open set $U_q \subset X$ such that whenever $p < q$, we have $\bar{U}_p \subset U_q$. Hint: enumerate all the rational numbers on the interval (so that the first two elements are 1 and 0) and then define $U_1 = X \setminus B$ and all other U_q's can be defined inductively by using normality of X.

Now let us extend the definition of U_q to all rational numbers by defining $U_q = \emptyset$ if q is negative, and $U_q = X$ if $q > 1$.

Next, for each $x \in X$ define $Q(x)$ to be the set of those rational numbers such that the corresponding set U_q contains x. Show that $Q(x)$ is bounded below and define $f(x)$ as its infimum.

Now we will show that $f(x)$ is the desired function. First, show that if $x \in \bar{U}_r$, then $f(x) \leq r$, and if $x \notin U_r$, then $f(x) \geq r$.

Now prove the continuity of $f(x)$ by showing that for any $x_0 \in X$ and an open interval (c, d) containing $f(x_0)$, there exist a neighbourhood U of x_0 such that $f(U) \subset (c, d)$. [Why would this imply continuity?] For this choose two rational numbers q_1 and q_2 such that $c < q_1 < f(x_0) < q_2 < d$ and take $U = U_{q_2} \setminus \bar{U}_{q_1}$.

Next, we will construct the metric space Y for the embedding. Actually, as a topological space, the space Y is simply the product of N copies of R with the product topology. Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the so-called standard bounded metric on R [show that this is indeed a metric]. Then if x and y are two points of Y , define

$$D(x, y) = \sup_i \bar{d}(x_i, y_i) .$$

Show that this is indeed a metric.

Proposition. The metric D induces the product topology on $Y = R^N$.

Proof. First, let U be open in the metric topology and let $x \in U$. We will find an open set V in the product topology such that $x \subset V \subset U$. Choose an ε-ball centered at x, which lies in U. Then choose N large enough so $1/N < \varepsilon$. Show that the following set satisfies the requirement:

$V = (x1 - \varepsilon, x1 + \varepsilon) \times \cdot\cdot\cdot \times (xN - \varepsilon, xN + \varepsilon) \times R \times R \times \cdot\cdot\cdot$.

Conversely, consider a basis element V = Q i∈N Vi for the product topology, such that Vi is open in R and Vi = R for all but finitely many indices i1, ..., iK. Given $x \in V$, we will find an open ball U in metric topology, which contains x and is contained in V . Choose an interval (xi − εi , xi + εi) contained in Vi such that $\varepsilon i < 1$ and define

$$\varepsilon = \min\{\varepsilon i/i \mid i = i1, ..., iK\}.$$

Now show that the ball of radius ε centered at x is contained in V .

Next we need the following technical result:

Lemma. Let X be a regular space with a countable basis. There exists a countable collection of continuous functions fn : X → [0, 1] such that for any $x0 \in X$ and any neighbourhood U of x0, there exists an index n such that $fn(x0) > 0$ and fn = 0 outside U.

Proof. Given x0 and U, use regularity to choose two open sets Bn and Bm from the countable basis containing x0 and contained in U such that B¯ n ⊂ Bm. Then use the Urysohn lemma to construct a function gn,m such that gn,m(B¯ n) = 1 and gn,m(X\Bm) = 0. Now show that this collection of functions satisfies our requirement.

Finally we will prove the main result:

Urysohn Metrization Theorem. Every regular space X with a countable basis is metrizable.

While every metrizable space is normal (and regular) such spaces do not need to be second countable. For example, any discrete space X is metrizable, but if X consists of uncountably many points it does not have a countable basis (Exercise 4.10). This means that the converse of the Urysohn Metrization Theorem does not hold. However, this theorem can be generalized to give conditions that are both sufficient and necessary for metrizability of a space. We finish this chapter by giving the statement of such result without proof.

Proof. Given the collection of functions {fn} from the previous lemma, and Y = R N with the product topology, we define a map F : X → Y as follows:

F(x) = (f1(x), f2(x), ...).

Show that this is a continuous map. Also show that it is injective.

In order to finish the proof, we need to show that for each open set U in X, the set F(U) is open in F(X). Let z0 be a point of F(U). Let x0 ∈ U be such that F(x0) = z0 and choose an index N such that fN (x0) > 0 and fN (X \ U) = 0.

Now we let W = π −1 N ((0,∞)) ∩ f(X),

where πN is the projection Y → R onto the Nth multiple. Show that W is an open subset of F(X) such that z0 ∈ W ⊂ F(U).

Give an example of a Hausdorff space with a countable basis which is not metrizable.

Urysohn Metrization Theorem. Every second countable normal space is metrizable. The main idea of the proof is to show that any space as in the theorem can be identified with a subspace of some metric space. To make this more precise we need the following:

Definition. A continuous function i: X → Y is an embedding if its restriction i: X → i(X) is a homeomorphism (where i(X) has the topology of a subspace of Y).

Example. The function i: (0, 1) → R given by i(x) = x is an embedding. The function j : (0, 1) → R given by j(x) = 2x is another embedding of the interval (0, 1) into R

Lemma. If j : X → Y is an embedding and Y is a metrizable space then X is also metrizable.

Proof. Let μ be a metric on Y . Define a metric ρ on X by ρ(x1, x2) = μ(j(x1), j(x2)). It is easy to check that the topology on X is induced by

Notes

the metric ρ (exercise). Let now X be a space as in Theorem 12.1. In order to show that X is metrizable it will be enough to construct an embedding j : X → Y where Y is metrizable. The space Y will be obtained as a product of topological spaces:

Definition. Let {Xi}i∈I be a family of topological spaces. The product topology on Q i∈I Xi is the topology generated by the basis

B = Q i∈I Ui | Ui is open in Xi and Ui 6= Xi for finitely many indices i only

Proposition. Let {Xi}i∈I be a family of topological spaces and for j ∈ I let

pj : Y i∈I Xi → Xj

be the projection onto the j-th factor: pj((xi)i∈I) = xj . Then:

1) for any j ∈ I the function pj is continuous.

2) A function f : Y → Q i∈I Xi is continuous if and only if the composition pjf : Y → Xj is continuous for all j ∈ I

Proof. Exercise Self

Definition. Let X be a topological space and let {fi}i∈I be a family of continuous functions fi : X → [0, 1]. We say that the family {fi}i∈I separates points from closed sets if for any point x0 ∈ X and any closed set A ⊆ X such that x0 6∈ A there is a function fj ∈ {fi}i∈I such that fj(x0) > 0 and fj |A = 0.

Embedding Lemma. Let X be a T1-space. If {fi : X → [0, 1]}i∈I is a family that separates points from closed sets then the map f∞ : X → Y i∈I [0, 1] given by

f∞(x) = (fi(x)) i∈I

is an embedding.

Note. If the family {fi}i∈I is infinitely countable then f∞ is an embedding of X into the Hilbert cube [0, 1]

Definition. Let X be a topological space. A collection U = {Ui}i∈I of open sets in X is locally finite if each point x ∈ X has an open neighborhood Vx such that Vx ∩ Ui 6= ∅ for finitely many i ∈ I only.

A collection U is countably locally finite if it can be decomposed into a countable union U = S∞ n=1 Un where each collection Un is locally finite.

Nagata-Smirnov Metrization Theorem. Let X be a topological space. The following conditions are equivalent:

1) X is metrizable.

2) X is regular and it has a basis which is countably locally finite.

Exercise. Show that the product topology on R n = R × · · · × R is the same as the topology induced by the Euclidean metric.

Exercise. Let {Xi}i∈I be a family of topological spaces. The box topology on Q i∈I Xi is the topology generated by the basis

$$B = Q \bigcup_{i \in i} Ui \mid Ui \text{ is open in } Xi$$

Notice that for products of finitely many spaces the box topology is the same as the product topology, but that it differs if we take infinite products.

Let X = Q∞ n=1[0, 1] be the product of countably many copies of the interval [0, 1]. Consider X as a topological space with the box topology. Show that the map f : [0, 1] → X given by f(t) = (t, t, t, . . .) is not continuous.

2 Urysohn's lemma

In topology, **Urysohn's lemma** is a lemma that states that a topological space is normal if and only if any two disjoint closed subsets can be separated by a continuous function.[1]

Urysohn's lemma is commonly used to construct continuous functions with various properties on normal spaces. It is widely applicable since all metric spaces and all compact Hausdorff spaces are normal. The lemma is generalized by (and usually used in the proof of) the Tietze extension theorem.

The lemma is named after the mathematician Pavel Samuilovich Urysohn.

10.2.1 Formal statement

Two subsets *A* and *B* of a topological space *X* are said to be separated by neighbourhoods if there are neighbourhoods *U* of *A* and *V* of *B* that are disjoint. In particular *A* and *B* are necessarily disjoint.

Two plain subsets *A* and *B* are said to be separated by a function if there exists a continuous function *f* from *X* into the unit interval [0,1] such that $f(a) = 0$ for all *a* in *A* and $f(b) = 1$ for all *b* in *B*. Any such function is called a **Urysohn function** for *A* and *B*. In particular *A* and *B* are necessarily disjoint.

It follows that if two subsets *A* and *B* are separated by a function then so are their closures.
Also it follows that if two subsets *A* and *B* are separated by a function then *A* and *B* are separated by neighbourhoods.

A normal space is a topological space in which any two disjoint closed sets can be separated by neighbourhoods. Urysohn's lemma states that a topological space is normal if and only if any two disjoint closed sets can be separated by a continuous function.

The sets *A* and *B* need not be precisely separated by *f*, i.e., we do not, and in general cannot, require that $f(x) \neq 0$ and $\neq 1$ for *x* outside of *A* and *B*. The spaces in which this property holds are the perfectly normal spaces.

Urysohn's lemma has led to the formulation of other topological properties such as the 'Tychonoff property' and 'completely Hausdorff spaces'. For example, a corollary of the lemma is that normal T_1 spaces are Tychonoff.

10.2.2 Sketch of Proof

The procedure is an entirely straightforward application of the definition of normality (once one draws some figures representing the first few steps in the induction described below to see what is going on), beginning with two disjoint closed sets. The *clever* part of the proof is the indexing the open sets thus constructed by dyadic fractions

For every dyadic fraction $r \in (0,1)$, we are going to construct an open subset $U(r)$ of X such that:

1. $U(r)$ contains A and is disjoint from B for all r
2. for $r < s$, the closure of $U(r)$ is contained in $U(s)$.

Once we have these sets, we define $f(x) = 1$ if $x \notin U(r)$ for any r; otherwise $f(x) = \inf \{ r : x \in U(r) \}$ for every $x \in X$. Using the fact that the dyadic rationals are dense, it is then not too hard to show that f is continuous and has the property $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.

In order to construct the sets $U(r)$, we actually do a little bit more: we construct sets $U(r)$ and $V(r)$ such that

- $A \subseteq U(r)$ and $B \subseteq V(r)$ for all r
- $U(r)$ and $V(r)$ are open and disjoint for all r
- for $r < s$, $V(s)$ is contained in the complement of $U(r)$ and the complement of $V(r)$ is contained in $U(s)$.

Since the complement of $V(r)$ is closed and contains $U(r)$, the latter condition then implies condition (2) from above.

This construction proceeds by mathematical induction. First define $U(1) = X \setminus B$ and $V(0) = X \setminus A$. Since X is normal, we can find two disjoint open sets $U(1/2)$ and $V(1/2)$ which contain A and B, respectively. Now assume that $n \geq 1$ and the sets $U(k/2^n)$ and $V(k/2^n)$ have already been constructed for $k = 1,...,2^n-1$. Since X is normal, for any $a \in \{ 0,1,...,2^n-1 \}$, we can find two disjoint open sets which contain $X \setminus V(a/2^n)$ and $X \setminus U((a+1)/2^n)$, respectively. Call these two open sets $U((2a+1)/2^{n+1})$ and $V((2a+1)/2^{n+1})$, and verify the above three conditions.

Notes

The Mizar project has completely formalized and automatically checked a proof of Urysohn's lemma in the URYSOHN3 file

Definition : An m-dimensional topological manifold is a Hausdor↲ space with a countable basis in which every point has a neighborhood homeomorphic to an open set in Rm.

Earlier in the semester, I left out the condition that manifolds should have a countable basis. The integer m is called the dimension of the manifold; it is uniquely determined by the manifold, because one can show that a nontrivial open subset in Rm can only be homeomorphic to an open subset in Rn when m = n. One-dimensional manifolds are called curves, two-dimensional manifolds are called surfaces; the study of special classes of manifolds is arguably the most important object of topology.

Of course, people were already studying manifolds long before the advent of topology; back then, the word "manifold" did not mean an abstract topological space with certain properties, but rather a submanifold of some Euclidean space. So the question naturally arises whether every abstractly defined manifold can actually be realized as a submanifold of some RN . The answer is yes; we shall prove a special case of this result, namely that every compact manifold can be embedded into RN for some large N.

Remark. One can ask the same question for manifolds with additional structure, such as smooth manifolds, Riemannian manifolds, complex manifolds, etc. This makes the embedding problem more dicult: for instance, John Nash (who was portrayed in the movie A Beautiful Mind) became famous for proving an embedding theorem for Riemannian manifolds.

Of course, every m-dimensional manifold can "locally" be embedded into Rm; the problem is how to patch these locally defined embeddings together to get a "global" embedding. This can be done with the help of the following tool.

Check In Progress

Q. 1 Give Introduction of Urysohn's Lemma.

Solution :

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Q. 2 Every regular space with a countable basis is normal.

Solution :

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10.3 EMBEDDING THEOREM: EMBEDDING OF TYCHONOFF SPACES IN RJ

Theorem (**Embedding Theorem**). Let X be a Tychonoff space. Suppose $(f_\alpha)_{\alpha\in J}$ is an indexed family of continuous function $f_\alpha : X$ cont $\longrightarrow \mathbb{R}$ satisfying the requirement that for each point $x_0 \in X$, each open nbhd U of x_0, $\exists \alpha \in J$ such that $f_\alpha(x_0) > 0$ and $\mathbb{R} \setminus U \subseteq f_\alpha^{-1}[\{0\}]$. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha\in J}$$

is an embedding of X into $\mathbb{R}^J$. If each f_α maps X into [0, 1], then F embeds in $[0, 1]^J$.

Proof. Replace n by α and RN by RJ throughout Step 2—notice that since we are assuming that we are given such a family, We need X to be T1 in order the guarantee that if x, y ∈ X and x 6= y, then there exists an index α such that fα(x) 6= fα(y), giving us an honest-to-God injection. Basically, the point is that this separates points from closed sets and we are able to make it separate points by demanding it be T1.

Remark. Such a family of functions is said to separate points from closed sets in X. (More generally, if for each closed set A, each x /∈ A, ∃f, f(x) ∈/ f[A].) If we only assumed X was T1, then the existence of such a family implies X is Tychonoff! This follows since RJ is Tychonoff for any J (a product of Hausdorff is Hausdorff and a product of completely regular spaces is completely regular and since complete regularity and Hausdorffness are hereditary), thus, it follows that the existence of such a collection implies X is Tychonof.

Corollary. If X is a T1 topological space and F ⊆ C(X, [0, 1]) separates points from closed sets, then e: X → [0, 1]F by πf (e(x)) = f(x) is an embedding and therefore X is Tychonoff.

10.4 EQUIVALENT CONDITION FOR A SPACE TO BE TYCHONOFF.

Theorem X is a Tychonoff space iff X is homeomorphic to a subspace of [0, 1]J for some set J.

Proof. (⇐=) Suppose X homeomorphic to a subspace of [0, 1]J for some set J. Since an arbitrary product of completely regular spaces is completely regular and since subspaces of completely regular spaces are completely regular, the homeomorphic image of X is completely regular. Since a product of T1 spaces is T1 and any subspace of a T1 space is T1, X is also, T1, hence, Tychonoff. (=⇒) Suppose X is Tychonoff. Let J index the collection of all continuous function from X into [0, 1]. For each x0 ∈ X and closed set A disjoint from x0, by complete regularity, there exists a continuous function fα : X → [0, 1] for some α ∈ J such that f(x0) = 1 and f[A] = {0}. But then if we replace A by the open set U

= X \ A, U is an open nbhd of x0 and f(x0) > 0 and f[X \ U] = {0}. Thus, the collection (fα)α∈J separates points from closed sets in X and thus the hypotheses of the embedding theorem and, hence, X is homeomorphic to a subspace of [0, 1]J .

10.5. SECOND COUNTABLE SPACE

Definition: The topological space (X,τ) is said to be **Second Countable** if there exists a basis B of τ that is countable.

Let (X,τ)(X,τ) be a topological space, then XX is said to be the second countable space, if ττ has a countable base.

In other words, a topological space (X,τ)(X,τ) is said to be the second countable space if it has a countable open base. A second countable space is also said to be a space satisfying the second axiom of countability. In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base. More explicitly, a topological space is second-countable if there exists some countable collection of open subsets of such that any open subset of can be written as a union of elements of some subfamily of . A second-countable space is said to satisfy the second axiom of countability. Like other countability axioms, the property of being second-countable restricts the number of open sets that a space can have.

Many "well-behaved" spaces in mathematics are second-countable. For example, Euclidean space (R^n) with its usual topology is second-countable. Although the usual base of open balls is uncountable, one can restrict to the collection of all open balls with rational radii and whose centers have rational coordinates. This restricted set is countable and still forms a basis.

10.5.1 Properties

Second-countability is a stronger notion than first-countability. A space is first-countable if each point has a countable local base. Given a base for a topology and a point *x*, the set of all basis sets containing *x* forms a local base at *x*. Thus, if one has a countable base for a topology then one has a countable local base at every point, and hence every second-countable space is also a first-countable space. However any uncountable discrete space is first-countable but not second-countable.

Second-countability implies certain other topological properties. Specifically, every second-countable space is separable (has a countable dense subset) and Lindelöf (every open cover has a countable subcover). The reverse implications do not hold. For example, the lower limit topology on the real line is first-countable, separable, and Lindelöf, but not second-countable. For metric spaces, however, the properties of being second-countable, separable, and Lindelöf are all equivalent. Therefore, the lower limit topology on the real line is not metrizable.

In second-countable spaces—as in metric spaces—compactness, sequential compactness, and countable compactness are all equivalent properties.

Urysohn's metrization theorem states that every second-countable, Hausdorff regular space is metrizable. It follows that every such space is completely normal as well as paracompact. Second-countability is therefore a rather restrictive property on a topological space, requiring only a separation axiom to imply metrizability.

10.5.2 Other Properties

- A continuous, open image of a second-countable space is second-countable.
- Every subspace of a second-countable space is second-countable.
- Quotients of second-countable spaces need not be second-countable; however, *open* quotients always are.
- Any countable product of a second-countable space is second-countable, although uncountable products need not be.
- The topology of a second-countable space has cardinality less than or equal to *c* (the cardinality of the continuum).

- Any base for a second-countable space has a countable subfamily which is still a base.
- Every collection of disjoint open sets in a second-countable space is countable.

Example: A topological space is termed separable if it admits a countable dense subset**.**

Proof

Given: A second-countable space X, with countable basis $\{B_n\}$

To prove: There exists a countable dense subset of X

Proof: We can assume without loss of generality that all the B_n are nonempty, because the empty ones can be discarded. Now, for each B_n, pick any element $x_n \in B_n$. Let D be the set of these x_n s. D is clearly countable (because the indexing set for its elements is countable). We claim that D is dense in X.

To see this, let U be any nonempty open subset of X. Then, U contains some B_n, and hence, $x_n \in U$. But by construction, $x_n \in D$, so D intersects U, proving that D is dense.

Theorem

Let T=(S,τ) be a second-countable topological space**.**

Then T is also a separable space**.**

Proof

By definition, there exists a countable basis B for τ.

Using the axiom of countable choice, we can obtain a choice function ϕ for B∖{∅}.

Define:

H={ϕ(B):B∈B∖{∅}}

By Image of Countable Set under Mapping is Countable, it follows that H is countable.

It suffices to show that H is everywhere dense in T.

Let x∈U∈.

By Equivalence of Definitions of Analytic Basis, there exists a B∈BB∈B such that x∈B⊆U.

Then ϕ(B)∈U, and so H∩U is non-empty.

Hence, xx is an adherent point of H.

By Equivalence of Definitions of Adherent Point, it follows that x∈H−, where H− denotes the closure of H.

Therefore, H−=S, and so H is everywhere dense in T by definition.

Check In Progress

Q. 1 Define Equivalent condition for Space.

Solution :

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Q. 2 State Embedding Theorem.

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105.3 Metrization Theorems

Introduction. What properties of a topological space (X, T) are enough to guarantee that the topology actually is given by some metric? The space has to be normal, since we know metric spaces are normal. And the topology has to have a countable local basis at each point, since metric spaces have that property. In Chapter 6 of the text, there are theorems saying, "(X, T) is metric if and only if it has the following topological properties . . . ". The conditions are (1) the space is regular, and (2) there is some countability property stronger than saying there is a countable local basis at each point, but a little weaker than 2nd-countable (but still strong enough that regular + the property =⇒ normal) . In the current text section, the theorem is less general: we characterize separable metric spaces; but this is a good introduction to the ideas.

One of the first widely recognized metrization theorems was **Urysohn's metrization theorem**. This states that every Hausdorff second-countable regular space is metrizable. So, for example, every second-countable manifold is metrizable. (Historical note: The form of the theorem shown here was in fact proved by Tychonoff in 1926. What Urysohn had shown, in a paper published posthumously in 1925, was that every second-countable normal Hausdorff space is metrizable). The converse does not hold: there exist metric spaces that are not second countable, for example, an uncountable set endowed with the discrete metric.[3] The Nagata–Smirnov metrization theorem, described below, provides a more specific theorem where the converse does hold.

Several other metrization theorems follow as simple corollaries to Urysohn's Theorem. For example, a compact Hausdorff space is metrizable if and only if it is second-countable.

Urysohn's Theorem can be restated as: A topological space is separable and metrizable if and only if it is regular, Hausdorff and second-countable. The Nagata–Smirnov metrization theorem extends this to the non-separable case. It states that a topological space is metrizable if and only if it is regular, Hausdorff and has a σ-locally finite base. A σ-locally finite base is a base which is a union of countably many locally finite collections of open sets. For a closely related theorem see the Bing metrization theorem.

Separable metrizable spaces can also be characterized as those spaces which are homeomorphic to a subspace of the Hilbert cube , i.e. the countably infinite product of the unit interval (with its natural subspace topology from the reals) with itself, endowed with the product topology.

Notes

A space is said to be **locally metrizable** if every point has a metrizable neighbourhood. Smirnov proved that a locally metrizable space is metrizable if and only if it is Hausdorff and paracompact. In particular, a manifold is metrizable if and only if it is paracompact.

How does one prove that some topology on a space is given by a metric? There are two choices: either explicitly define the metric and prove the metric topology is the same as T , or show that X is homeomorphic to a subspace of a known metric space. We used the first approach when we defined a metric on R^ω that generates the product topology; and now we will see a good example of the other approach, which is also how the most general metrization theorems are proven.

Theorem (Urysohn metrization theorem). If (X, T) is a regular space with a countable basis for the topology, then X is homeomorphic to a subspace of the metric space R^ω

The way I stated the above theorem, it is ambiguous: we have studied two (inequivalent) metrics for R^ω : the product space metric and the uniform metric. The theorem is true with either metric, but it is an "if and only if" for the product metric. Recall that in the product topology, R^ω has a countable dense subset: the set S = all vectors $(q_1, q_2, \ldots)$ where each $q_i \in Q$ and all but finitely many q_i are 0. Since R^ω in the product topology is metrizable and has a countable dense subset, it must be 2nd-countable. Each subspace of space with a countable basis also has a countable basis. And, of course, each subspace of a metric space is metric. We conclude that each subspace of (R^ω , product metric) is metric and has a countable basis.

The above paragraph combines ideas from various parts of our course. So let us take this as one (bunch of) of our sample problems for the Final Exam. Your task s to organize the various facts, ideas, etc. into a coherent proof (and to be able to fill in details if asked); as always, an exam problem might involve filling in the details of one particular part of a longer argument).

Problem.

If (X, T) is homeomorphic to a subspace of (R ω , product topology), then (X, T) is regular and has a countable basis.

Key steps:

a. There is a metric on R ω that gives the product topology.

b. R ω in the product topology has a countable dense subset.

c. A metric space with a countable dense subset has a countable basis for the topology.

d. Each subspace of a 2nd-countable space is 2nd-countable. e. Each subspace of a metric space is metrizable.

e. Each metric space is regular. f. Put the pieces together.

On the other hand, the metric space (R ω , uniform metric) is not second-countable (so not separable) since it has an uncountable discrete subspace K = the set of all vectors (t1, t2, . . .) where each ti = 0 or 1. The set of all sequences of 0's and 1's is uncountable, and the distance between any two elements of K is 1. So each subspace of (R ω , uniform metric topology) is a metric space, but it need not be separable. We really should state the Urysohn metrization theorem as two theorems:

Theorem. (X, T) is regular with a countable basis ⇐⇒ (X, T) is homeomorphic to a subspace of (R ω , product topology metric).

Theorem. (X, T) is regular with a countable basis =⇒ (X, T) is homeomorphic to a subspace of (R ω , uniform metric topology).

Proving the metrization theorem[s]. The text gives the details, so I will focus on the gestalt and some highlights. Our goal is to define an embedding of X into R ω . We want to assign to each point x ∈ X a point F(x) ∈ R ω , that is a (countably infinite) list of "coordinates": F(x) = (x1, x2, . . .). How can we find numbers that measure how a point x ∈ X is related topologically to all the other points of X? This is the bit of magic in this theorem. We will use Urysohn's lemma infinitely many times to define a sequence of functions fn : X → [0, 1]; these will be the coordinate functions.

The space (X, T) has a countable basis B and it it regular, so it is normal. Given any closed set A and open neighborhood U(A), there exists a

Notes

Urysohn function for the disjoint closed sets X − U and A. That is, there exists f : X → [0, 1] such that f(x) = 0 for all x /∈ U and f(a) = 1 for all a ∈ A. In particular, for any pair Bn, Bm of elements of B that happen to have B¯ n ⊆ Bm, there exists a function f : X → [0, 1] with f = 1 on B¯ n and f = 0 outside Bm.

Since B is countable, the set of such pairs Bn, Bm is countable. Number these pairs in any order, and let f1, f2, . . . be the Urysohn functions defined in the preceding paragraph. Then define F : X → [0, 1] by

F(x) = (f1(x), f2(x), . . .).

We need to prove that the function F is 1-1, continuous, and has a continuous inverse (From F(X) → X). The questions of continuity have to depend on what topology we use for R ω . But we can check injectivity before worrying about the topology

Proposition. The function F : X → R ω is injective

Proof. Suppose x, y ∈ X with x 6= y. Since X is Hausdorff, there exist disjoint neighborhoods U(x), V (y). Since B is a basis, there exists some Bm with x ∈ Bm ⊆ U. Since X is regular, there exists a neighborhood U ′ (x) such that U¯′ ⊆ Bm. And, again since B is a basis, there exists a basis set Bn with x ∈ Bn ⊆ U ′ . Since U¯′ ⊆ Bm, we thus have B¯ n ⊆ Bm. The Urysohn function fj associated to the pair Bn, Bm has x → 1 and y → 0; so F(x) 6= F(y).

The text goes on to show that, in the product topology, F is continuous and has a continuous inverse. The proof that F is continuous is easy because each coordinate function is continuous; the proof that F is an open map takes more work; see the text for the details.

To use the uniform topology, we need to change F. Recall that in the product topology, if we are studying a function from a space into a product space, i.e. some G : Y → Q α∈J Xα, and we want to show that G is continuous, it is sufficient to check that each component function Gα : Y → Xα is continuous. But in the uniform topology, this is not sufficient.

Example (Page 127, problem 4a). The function G : R → R ω defined by G(t) = (t, 2t, 3t, 4t, . . .) is not continuous in the uniform topology on R ω

. In particular, there is no neighborhood of 0 that is mapped by G into a uniform ϙ-neighborhood of (0, 0, 0, . . .).

To make the coordinate function F topologically "well-behaved" for the uniform metric on R ω , we need to eliminate the difficulty suggested by the above example. We do this by making the coordinate functions fj get smaller as j gets larger. Specifically, define

G : X → R ω by G(x) = (f1(x), 1 2 f2(x), 1 3 f3(x), 1 4 f4(x), . . .)

There is one more part of the proofs that is "cute", "clever", or 'annoyingly slick", depending on your tastes: The uniform metric topology is finer than the product topology on R ω . SO once we know F is an open map in the product topology (that takes work), it is easy to see that F, hence G, is an open map in the uniform topology. Conversely, once we know G is continuous in the uniform topology (that takes work), it is easy to see that F is continuous in the product topology.

Here are some sample problems for the Final Exam that you can use to solidify your understanding of these proofs. The first is an easier special case; the others are the "standard" Urysohn metrization theorem.

Problem:

1. Prove: If X is a compact Hausdorff space with a countable basis, then there exists an embedding of X into R ω , where R ω has the product topology.
2. Write a one-to-two page proof:
   If X is a regular space with a countable basis, then there exists an embedding of X into R ω , where R ω has the uniform topology.
3. Write a one-to-two page proof:
   If X is a regular space with a countable basis, then there exists an embedding of X into R ω , where R ω has the unifom topology.
4. If XX is finite, then a member of each ττon XX is finite. So its base is finite. Hence (X,τ)(X,τ) is the second countable space. Now we show that (X,τ)(X,τ) is the first countable space. Let SS be a subbase of ττ. So, S⊆P(X)S⊆P(X) (Countable), then B⊆P(X)B⊆P(X) (countable), so BB is also countable.

Therefore, (X,τ)(X,τ) is the second countable space, as each local base is also countable, so this is also the first countable space.

5. Consider the disjoint countable union . Define an equivalence relation and a quotient topology by identifying the left ends of the intervals - that is, identify 0 ~ 2 ~ 4 ~ … ~ 2k and so on. *X* is second-countable, as a countable union of second-countable spaces. However, *X*/~ is not first-countable at the coset of the identified points and hence also not second-countable.
6. The above space is not homeomorphic to the same set of equivalence classes endowed with the obvious metric: i.e. regular Euclidean distance for two points in the same interval, and the sum of the distances to the left hand point for points not in the same interval -- yielding a strictly weaker topology than the above space. It is a separable metric space (consider the set of rational points), and hence is second-countable.
7. The long line is not second-countable, but it is first-countable.

10.6 SUMMARY

We learnt in this unit Embedding theorem and its properties. We study Sketch and its properties with some examples. We study Metrization theorem.

10.7 KEYWORD

Embedding : Attach (a journalist) to a military unit during a conflict

Matrization : In topology and related areas of mathematics, a metrizable space is a topological space that is .... of a topological space of being homeomorphic to a uniform space, or equivalently the topology being ***defined*** by a family of pseudometrics .

Sketch : A short humorous play or performance, consisting typically of one scene in a revue or comedy programme

10.8 QUESTIONS FOR REVIEW

1. Let X1, X2 be topological spaces, and for i = 1, 2 let pi : X1 × X2 → Xi be the projection map.

   a) Show that if a set U ⊆ X1 × X2 is open in X1 × X2 then pi(U) is open in Xi .

   b) Is it true that if A ⊆ X1 × X2 is a closed set then pi(A) must be closed is Xi ?

   Justify your answer

2. Let X, Y be topological spaces. For a (not necessarily continuous) function f : X → Y the graph of f is the subspace Γ(f) of X × Y given by

   Γ(f) = {(x, f(x)) ∈ X × Y | x ∈ X}

   Show that if Y is a Hausdorff space and f : X → Y is a continuous function then Γ(f) is closed in X × Y .

3. Assume that X, Y are spaces such that R ∼= X × Y . Show that either X or Y is consists of only one point.

4. Let X and Y be non-empty topological spaces. Show that the space X × Y is connected if and only if X and Y are connected.

5. Let {Xi}i∈I be a family of topological spaces and for i ∈ I let Ai be a closed set in Xi . Show that the set Q i∈I Ai is closed in the product topology on Q i∈I Xi .

6. Show that the product topology on R n = R × · · · × R is the same as the topology induced by the Euclidean metric.

7. If j : X → Y is an embedding and Y is a metrizable space then X is also metrizable.

8. Every second countable space is the first countable space, but the converse may not be true.

9. Any uncountable set X with a co-finite topology is not the first countable space and so it is not the second countable space.

10.9 SUGGESTION READING AND REFERENCES

Willard, Stephen (1970). *General Topology*. Dover Publications. ISBN 0-486-43479-6.

- Stephen Willard, *General Topology*, (1970) Addison-Wesley Publishing Company, Reading Massachusetts.
- John G. Hocking and Gail S. Young (1961). *Topology*. Corrected reprint, Dover, 1988. ISBN 0-486-65676-4

10.10 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1

Q 2 Check in Section 2

Check in Progress-II

Answer Q. 1 Check in Section 3

Q 2 Check in Section 4

UNIT 11: CONNECTED & PATH-CONNECTED SPACE

STRUCTURE

11.0 OBJECTIVE

* Learn Connected and path connected space

* Learn Connectedness and Sepration

* Learn Closure of Connected Sets

* Learn diff. b/w connected and path connected

11.1 INTRODUCTION

In topology and related branches of mathematics, a **connected space** is a topological space that cannot be represented as the union of two or more disjoint non-empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces.

A subset of a topological space *X* is a **connected set** if it is a connected space when viewed as a subspace of *X*.

Some related but stronger conditions are path connected, simply connected, and n-connected. Another related notion is locally connected, which neither implies nor follows from connectedness.

11.1.1 Formal definition

A topological space *X* is said to be **disconnected** if it is the union of two disjoint non-empty open sets. Otherwise, *X* is said to be **connected**. A subset of a topological space is said to be connected if it is connected under its subspace topology. Some authors exclude the empty set (with its unique topology) as a connected space, but this article does not follow that practice.

For a topological space X the following conditions are equivalent:

1. X is connected, that is, it cannot be divided into two disjoint non-empty open sets.
2. X cannot be divided into two disjoint non-empty closed sets.
3. The only subsets of X which are both open and closed (clopen sets) are X and the empty set.
4. The only subsets of X with empty boundary are X and the empty set.
5. X cannot be written as the union of two non-empty separated sets (sets for which each is disjoint from the other's closure).
6. All continuous functions from X to $\{0,1\}$ are constant, where $\{0,1\}$ is the two-point space endowed with the discrete topology.

11.1.2 Connected Components

The maximal connected subsets (ordered by inclusion) of a non-empty topological space are called the **connected components** of the space. The components of any topological space X form a partition of X: they are disjoint, non-empty, and their union is the whole space. Every component is a closed subset of the original space. It follows that, in the case where their number is finite, each component is also an open subset. However, if their number is infinite, this might not be the case; for instance, the connected components of the set of the rational numbers are the one-point sets (singletons), which are not open.

Let be the connected component of x in a topological space X, and be the intersection of all clopen sets containing x (called quasi-component of x.) Then where the equality holds if X is compact Hausdorff or locally connected.

11.1.3 Disconnected Spaces

A space in which all components are one-point sets is called totally disconnected. Related to this property, a space X is called **totally separated** if, for any two distinct elements x and y of X, there exist disjoint open sets *U* containing *x* and *V* containing *y* such that *X* is the union of *U* and *V*. Clearly, any totally separated space is totally disconnected, but the converse does not hold. For example take two copies of the rational numbers **Q**, and identify them at every point except zero. The resulting space, with the quotient topology, is totally disconnected. However, by considering the two copies of zero, one sees that the space is not totally separated. In fact, it is not even Hausdorff, and the condition of being totally separated is strictly stronger than the condition of being Hausdorff.

Examples

- The closed interval [0, 2] in the standard subspace topology is connected; although it can, for example, be written as the union of [0, 1) and [1, 2], the second set is not open in the chosen topology of [0, 2].
- The union of [0, 1) and (1, 2] is disconnected; both of these intervals are open in the standard topological space [0, 1) ∪ (1, 2].
- (0, 1) ∪ {3} is disconnected.
- A convex subset of $\mathbf{R}^n$ is connected; it is actually simply connected.
- A Euclidean plane excluding the origin, (0, 0), is connected, but is not simply connected. The three-dimensional Euclidean space without the origin is connected, and even simply connected. In contrast, the one-dimensional Euclidean space without the origin is not connected.
- A Euclidean plane with a straight line removed is not connected since it consists of two half-planes.
- $\mathbb{R}$, The space of real numbers with the usual topology, is connected.
- If even a single point is removed from $\mathbb{R}$, the remainder is disconnected. However, if even a countable infinity of points are removed from , where $n \geq 2$, the remainder is connected. If $n \geq 3$, then remains simply connected after removal of countable many points.
- Any topological vector space, e.g. any Hilbert space or Banach space, over a connected field (such as or), is simply connected.

- Every discrete topological space with at least two elements is disconnected, in fact such a space is totally disconnected. The simplest example is the discrete two-point space.[1]
- On the other hand, a finite set might be connected. For example, the spectrum of a discrete valuation ring consists of two points and is connected. It is an example of a Sierpiński space.
- The Cantor set is totally disconnected; since the set contains uncountably many points, it has uncountably many components.
- If a space *X* is homotopy equivalent to a connected space, then *X* is itself connected.
- The topologist's sine curve is an example of a set that is connected but is neither path connected nor locally connected.
- The general linear group (that is, the group of *n*-by-*n* real, invertible matrices) consists of two connected components: the one with matrices of positive determinant and the other of negative determinant. In particular, it is not connected. In contrast, is connected. More generally, the set of invertible bounded operators on a complex Hilbert space is connected.
- The spectra of commutative local ring and integral domains are connected. More generally, the following are equivalent[2]
    1. The spectrum of a commutative ring *R* is connected
    2. Every finitely generated projective module over *R* has constant rank.
    3. *R* has no idempotent (i.e., *R* is not a product of two rings in a nontrivial way).

An example of a space that is not connected is a plane with an infinite line deleted from it. Other examples of disconnected spaces (that is, spaces which are not connected) include the plane with an annulus removed, as well as the union of two disjoint closed disks, where all examples of this paragraph bear the subspace topology induced by two-dimensional Euclidean space.

11.2 CONNECTEDNESS AND SEPARATIONS

Definition (Separation & Connected Spaces). Let X be a topological space. A separation of X is a pair U, V of nonempty, disjoint open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Remark. In the above definition, notice that if X can be separated by nonempty sets U and V , then U ∪· V = X so that X \ U = V . That is, U c = V where the complement is understood to be with respect to the space X.

Proposition . A space X is connected iff the only subsets of X that are both open and closed in X are Ø and X itself.

Proof. We prove the equivalent statement: A space X is not connected iff there exists a nonempty proper subset of X that is clopen. (⇐=) If A ⊂ X is a proper subset of X that is clopen in X, then the sets U = A and V = X \A constitute a separation of X for they are nonempty, disjoint open sets and their union is X. (=⇒) Conversely, if X is not connected, then it has a separation—say U and V form a separation of X. Then U 6= X is nonempty and U c = V is open, so U is clopen. Since V = X \ U, we're done.

11.2.1 A Separation (A, B) of a Subspace Y ⊆ X Satisfies A ∩ B = A ∩ B = Ø in X.

For a subspace Y of a topological space X, there is another useful way of formulating the definition of connectedness:

Lemma 6. Let Y be subspace of X. Then the sets A and B form a separation of Y iff A and B are a pair of disjoint nonempty sets whose union is Y and neither of which contain a limit point of the other (i.e., A ∩ B = A ∩ B = Ø in X). If there exists no separation of Y , then Y is connected.

Proof. (=⇒) Suppose A and B separate Y . Then A, B 6= Ø, are open and disjoint subsets of Y such that A ∪· B = Y . By Theorem 1.2, ClY

(A) = ClX(A) ∩ Y and since A is closed in Y , A = ClX(A) ∩ Y . But then ClX(A) ∩ B = Ø. Since ClX(A) is the union of A and its limit points, this means that B contains no limits points of A. Since B = Ac , the same reasoning shows that A∩ClX(B) = Ø. (⇐=) Suppose A and B are disjoint, nonempty sets whose union is Y and neither of which contains a limit point of the other. Letting bars denote closures in X, then A ∩ B = A ∩ B = Ø; therefore, we conclude that A ∩ Y = A and B ∩ Y = B. Thus, both A and B are closed in Y and since A = Y \ B and B = Y \ A, it follows that A and B are both open in Y as well.

11.2.2 Separations Absorb Connected Subspaces.

Lemma 7. If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof. Since C and D are both open in X, C ∩ Y and D ∩ Y are open in Y . These two sets are disjoint and their union is Y ; if they were both nonempty, they would constitute a separation of Y . Therefore, one of them is empty. Hence, Y must lie entirely within either C or D.

11.2.3 Condition for a Union of Connected Sets to be Connected.

Theorem : If {Cα}α∈A is a family of connected subspaces of X such that T α∈A Cα 6= Ø, then S α∈A Cα is connected. Proof. Let p ∈ T α∈A Cα and put Y = S α∈A Cα. Suppose on the contrary that Y = C ∪· D is a separation of Y . Then either p ∈ C or p ∈ D. Suppose WLOG p ∈ C. Since Cα is connected, it must lie entirely in either C or D by the lemma, and it cannot lie in D since p ∈ C ∩ Cα. Hence, Cα ⊆ C for every α ∈ A so that Y = S α∈A Cα ⊆ C, contradicting the fact D 6= Ø.

REMARK. WE SHALL GENERALIZE THIS THEOREM IN THE EXERCISES.

Check In Progress

Q. 1 Define Connected Space.

Notes

Solution :

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Q. 2 Define Disconnected Space.

Solution :

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11.3 CLOSURES OF CONNECTED SETS.

Theorem 6.29. Let A be a connected subspace of X. If A ⊆ B ⊆ A, then B is also connected.

Proof. Suppose B = C ∪· D is a separation of B. By the above lemma, A must lie entirely in C or in D; suppose that A ⊆ C WLOG. Then A ⊆ C. By Lemma 6, C ∩ D = Ø, so B cannot intersect D. This contradicts the assumption that D 6= Ø.

Corollary 4. Let A be a connected subspace of X. Then A is connected.

Proof. This is a special case of the above more general theorem

11.3.1 Continuous Image of a Connected Space is Connected.

Theorem 6.30. The image of a connected space under a continuous map is connected.

Proof. Let f : X cont −→ Y ; let X be connected. We wish to show that Z = f[X] is connected. Since the map obtained from f by restricting its range is also continuous, it suffices to consider the case that f is surjective as well. Suppose that Y = A ∪· B is a separation of Y . Then f −1 [A], and f −1 [B] are open by continuity, clearly disjoint and nonempty since f is surjective and hence comprise a separation of X, contradicting the assumption that X was connected.

Corollary 5 (**Invariance of Connectedness Under Homeomorphism**). If Y is connected and f : X → Y is a homeomorphism, then X is connected.

Proof. f −1 : Y → X is a continuous surjective map and hence by the theorem, f −1 [Y] = X is connected.

11.3.2 A Useful Equivalent Definition of A Connected Set.

Definition **(Totally Disconnected Space**). A totally disconnected space is one in which the only nonempty connected sets are singletons.

Remark. Totally Disconnected spaces exist. For instance, Z ⊆ R with the subspace topology or even {0, 1} ⊆ R with the subspace topology.

Proposition 20. A space X is connected iff there is a totally disconnected space Y of cardinality greater than one such that every continuous function f : X → Y is constant.

Proof. (=⇒) Fix Y a totally disconnected space with #(Y) > 1. Suppose X is connected. If X = Ø, this is vacuously true so suppose X 6= Ø. By the theorem above, if f : X → Y is continuous, then f[X] is connected and nonempty and, hence, must be a singleton. Since Y was arbitrary, we’re done. (⇐=) WLOG fix Y a totally disconnected space of cardinality greater than one. Suppose for the sake of a contradiction that every continuous function f : X → Y is constant but that X is not connected. Let U, V be a separation of X and define g : X → Y by g(x) = y0 and g(x) = y1. We claim g is continuous. To see this, note that g −1 [{y0}] = U which is open and nonempty, g −1 [{y1}] = V which is open and nonempty and g −1 [Y] = X. Hence, it is clear that for every open subset B ⊆ Y , g −1 [B] is open so that g is continuous, contradicting the assumption that every continuous function is constant. Since Y was arbitrary, we're done.

Remark. The proof of the above proposition reveals that the choice of a totally disconnected space Y of cardinality greater than one does not matter.

Hence, we might rephrase the above proposition as follows:

Proposition 21. A space X is connected iff every continuous function f : X → {0, 1} is constant.

Proof. Virtually the same as above.

Thus, in actual practice we use this second formulation, where we put Y = {0, 1} a subspace R. This makes things nice.

11.3.3 A Finite Products of Connected Spaces is Connected.

Theorem 6.31. A finite product of spaces is connected iff each space is connected.

Proof. (=⇒) Suppose X = Qn i=1 Xi is connected. This direction is trivial as the projection maps πα : X → Xα are continuous and surjective, so that πα[X] = Xα forcing Xα to be connected. (⇐=) This is the meat of the problem. The proof is by induction. Since the induction step is clear, we prove the base case of two connected spaces, say X and Y . The idea is to write the space as a union of connected spaces all sharing a common point so that we may apply Theorem 1.28—what we shall do in this proof is move "vertical slices" along a "horizontal slice".

Towards this end, choose a “base point” $(a, b) \in X \times Y$. Note that the “horizontal slice” $X \times \{b\}$ is connected being homeomorphic to X and each “vertical slice” $\{x\} \times Y$ is connected being homeomorphic with Y . Hence, each “T-shaped” space.

$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

is connected by as it is union of two connected spaces that have the point (x, b) in common. Now the union X × Y = S x∈X Tx of all these T-shaped spaces is connected because it is a union of connected spaces such that T x∈X Tx 6= Ø since (a, b) ∈ T x∈X Tx. Hence, X × Y is connected.

Remark. It is natural to ask whether this theorem extends to arbitrary products of connected spaces. The answer depends on which topology is used for the product. For example, if we suppose for the moment that we know R is connected, then RN is not connected the box topology but it is connected in the product topology. Indeed, a arbitrary product of connected spaces is connected in the product topology and we shall prove this in an exercise.

Example . Let RN have the box topology. We can write RN as the union of the set A consisting of all bounded sequences and B consisting of all unbounded sequences. These sets are disjoint, and each is open in the box topology, for if a ∈ RN, the open set U = Q i∈N(ai − 1, ai + 1) consists entirely of bounded sequences if a is bounded and unbounded sequences if a is unbounded. Thus, even though R is connected (as we shall prove), RN is not connected in the box topology.

11.4 PATH CONNECTED

Definition (Paths & Path-Connected Spaces). Given points x and y of the space X, a path in X from x to y is a continuous map $f : [0, 1] \to X$ (or $[a, b] \subseteq R$) such that $f(0) = x$ and $f(1) = y$. A space X is said to be path-connected if every pair of points of X can be joined by a path in X.

Proposition (**Path Connected Spaces are Connected**). Let X be a path connected space. Then X is connected.

Notes

Proof. Suppose for the sake of a contradiction that X has a separation A, B. Let $a \in A$ and $b \in B$ and let $f : [0, 1] \to X$ be a path in X from a to b. Since we have seen [a, b] is connected and that continuous images of connected sets, it follows that f [[0, 1]] is a connected subspace of X and, hence, lies entirely in A or entirely in B. But then it is impossible that $f(0) = a$ and $f(1) = b$, a contradiction.

A Connected Space That is Not Path-Connected.

Example . Let $S = \{(x,\sin(1/x)) : 0 < x \leq 1\} \subseteq R$ be a subspace. Because S is the image of the connected set (0, 1] under a continuous map, S is connected and hence its closure S in R2 is connected as well. We call S the topologist's sine curve; $S = (\{0\} \times [-1, 1]) \cup\cdot S$. We shall show that S is not connected. Suppose $f : [0, 1] \to S$ is a path beginning at the origin and ending at a point of S. The set $f^{-1}[\{0\} \times [-1, 1]]$ is closed as f is continuous and $\{0\} \times [-1, 1]$ is closed in R2 and hence S and it is bounded as it is a subset of [0, 1]. Hence, putting $\alpha = \sup f^{-1}[\{0\} \times [-1, 1]]$, we have $\alpha \in f^{-1}[\{0\} \times [-1, 1]]$. Then the restriction $f : [\alpha, 1] \to S$ is a path that maps α into $\{0\} \times [-1, 1]$ and maps (b, c] into S. Write f(t) = (xf (t), yf (t)) and note that for t > 0, yf (t) = sin(1/xf (t)). Given $n \in N$, choose $u \in R$ such that 0 < u < xf (1/n)) and such that sin(1/u) = (−1)n. By the IVT there exists $tn \in R$ with 0 < tn < 1/n such that x(tn) = u. Repeat this process for all $n \in N$ and consider the sequence (tn). Then tn → 0 while yf (tn) could not possibly converge, contradicting the assumption of continuity of f.

Remark. It is also useful to "loop" S back in on itself (don't take the closure, connect back up with (1,sin(1))) to create a space that is useful for some other counter examples.

Path-Connected Spaces Satisfy Some Analogous Properties of Connected Spaces.

While the topologist's sine curve shows that the closure of path-connected space need not be path-connected, it turns out that path-connected spaces enjoy other analogous properties of connected spaces.

11.4.1 The Principle of Transfinite Induction.

Reminder. Let (S, ≤) be a poset. Zorn's lemma states that if every non-empty, totally ordered subset J ⊆ S has an upper bound in S (i.e., an element α ∈ S such that ∀j ∈ J, j ≤ α), then S contains a maximal element—that is, an element m such that for any x ∈ S, if x ≥ m, then x = m.

Reminder (Inductive Order). Let S be a set and ≤ be a partial order on S. We say that ≤ is an inductive order on S if every non-empty, totally ordered subset J of S has an upper bound in S. In other words, an inductive order satisfies the hypotheses of Zorn's lemma. We also say that a set S is inductively ordered by a partial order ≤ if it satisfies the hypotheses of Zorn's lemma.

First let us prove the principle of transfinite induction

Exercise. Let J be a set and < be a well-order on J. For any set J0 ⊆ J satisfying the property that for any α ∈ J, if {x ∈ J : x < α} ⊆ J0, then α ∈ J, we have J0 = J.

Proof. WLOG J 6= Ø. Let β be the minimal element of J under the well-order < β} = Ø and certainly Ø ⊆ J0. For each α ∈ J, let Sα = {x ∈ J : x < α} and let T = {α ∈ J : Sα ⊆ J0}. Since β has an immediate successor, say β + 1, Sβ+1 ⊆ J0 so that T 6= Ø. If T 6= J, then there exists an element x1 ∈ J \ T that is minimal among all elements in J \ T . Let x < x1 be a predecessor. Then Sx ⊆ J0 as otherwise x ∈ J \ T which is impossible by minimality of x1. Hence, for each x < x1, Sx ⊆ J0 and hence x ∈ J0. Thus, {x ∈ J : x < x1} ⊆ J0 and hence x1 ∈ T , contradicting the assumption that x1 ∈ J \ T . Hence, J = T . But then J = S α∈J Sα ⊆ J0 and hence as J ⊇ J0 already, we must have J = J0, as desired.

Remark. It therefore follows that if P is a property on a well-ordered set J with minimal element 0 and P satisfies the property that P(0) is true and for every α > 0, if P(y) is true for all y < α, then P(α) is true, then P(x) is true for every x ∈ J.

11.4.2 The Long Line.

We might need to go to the index to refresh on these concepts. Recall that SΩ is the minimal uncountable well-ordered set—it is unique up to order type. We typically write SΩ = [0, Ω) or if there could be any confusion, SΩ = [a0, Ω) where 0 (or a0) is the smallest element of SΩ as guaranteed by well-ordering and by design Ω is the largest element not in SΩ and is such that if α < Ω, then Sα is countable (we obtain this by the imposition of a well-ordering on an uncountable set). Let L denote the set SΩ × [0, 1) in the lexicographic order (i.e., the dictionary order) with its smallest element deleted (i.e., (0, 0)) and with topology induced by this total ordering. The space L def = (0, Ω) × (0, 1] equipped with the order topology described is called the long line.

Exercise. The long line is path connected and locally homeomorphic to R, but cannot be embedded in Rd for any d ∈ N.

(Step 1.) Let X be an ordered set; let a < b < c be points of X. Then [a, c) has the order type of [0, 1) iff [a, b) and [b, c) have the order type of [0, 1).

(Step 2.) Let X be an ordered set and x0 < x1 < · · · an increasing sequence of points in X; suppose b = sup{xi}. Then [x0, b) has the order type of [0, 1) iff each interval [xi , xi+1) has the order type of [0, 1).

(Step 3.) Let a0 denote the smallest element of SΩ. For each element a ∈ SΩ \ {a0}, show that [a0 × 0, a × 0) of SΩ × [0, 1) has the order type of [0, 1). [Hint: Proceed by transfinite induction. Either a has an immediate predecessor in SΩ or there is an increasing sequence (ai) in SΩ with a = sup{ai}.]

(Step 4.) L is path-connected.

(Step 5.) Every point of L has a nbhd homeomorphic with an open interval in R—thus, L is locally Euclidean.

(Step 6.) L cannot be embedded in Rd for any d ∈ N. [Hint: Any subspace of Rd is second-countable.]

Proof. Let us prove steps (1), (2) and (3) as “lemmas”, first.

(Step 1.) Let X be an ordered step and a < b < c be points of X. (=⇒) Suppose [a, c) has the order type of [0, 1)—that is, there is an order-preserving bijection between them—we wish to prove that [a, b) and [b, c) have the order type of [0, 1). Let ϕ: [a, c) → [0, 1) be an order-preserving bijection; then ϕ embeds [a, b) and [b, c) in [0, 1) in an order-preserving manner. Now, we contend that ϕ[[a, c)] and ϕ[[b, c)] are intervals in [0, 1). If ϕ[[a, c)] ⊆ [0, 1) were not an interval, then there exists an element x ∈ [0, 1) such that ϕ(a) < x < ϕ(c) but x /∈ ϕ[[a, c)] which is impossible, clearly, by virtue of our assumption on ϕ—a similar argument works for [b, c). We may write ϕ[[a, b)] = [ϕ(a), ϕ(b)) since ϕ must map a to 0 and since ϕ[[a, b)] ∩ ϕ[[b, c)] = Ø so that ϕ(b) ∈/ ϕ[[a, b)]. This is clear. It therefore suffices to show that there is an order-preserving bijection between an interval [α, β) ⊆ [0, 1) with [0, 1) where β > α. It is clear that we may assume α = 0 WLOG. Thus, we must show that there is an order-preserving bijection between [0, β) ⊆ [0, 1) and [0, 1). This is trivial: Define ψ: [0, β) → [0, 1) by ψ(x) = x/β. This is an injective linear map and thus is order-preserving and is a bijection because ψ(0) = 0 and ψ(1) = 1. (⇐=) Suppose ϕ1 : [a, b) → [0, 1) is an ordering-preserving bijection and ϕ2 : [b, c) → [0, 1) is an order-preserving bijection. Define ψ: [a, c) → [0, 1) by

this is obviously an order-preserving bijection.

11.5 PATH CONNECTEDNESS

A **path-connected space** is a stronger notion of connectedness, requiring the structure of a path. A path from a point *x* to a point *y* in a topological space *X* is a continuous function *f* from the unit interval [0,1] to *X* with *f*(0) = *x* and *f*(1) = *y*. A **path-component** of *X* is an equivalence class of *X* under the equivalence relation which makes *x* equivalent to *y* if there is a path from *x* to *y*. The space *X* is said to be **path-connected** (or **pathwise connected** or **0-connected**) if there is exactly one path-component, i.e. if there is a path joining any two points in *X*. Again, many authors exclude the empty space.

Every path-connected space is connected. The converse is not always true: examples of connected spaces that are not path-connected include the extended long line L^* and the topologist's sine curve.

Subsets of the real line **R** are connected if and only if they are path-connected; these subsets are the intervals of **R**. Also, open subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are connected if and only if they are path-connected. Additionally, connectedness and path-connectedness are the same for finite topological spaces.

Check In Progress

Q. 1 Let J be a set and < be a well-order on J. For any set J0 ⊆ J satisfying the property that for any α ∈ J, if {x ∈ J : x < α} ⊆ J0, then α ∈ J, we have J0 = J.

Solution

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Q. 2 Define Long Line.

Solution

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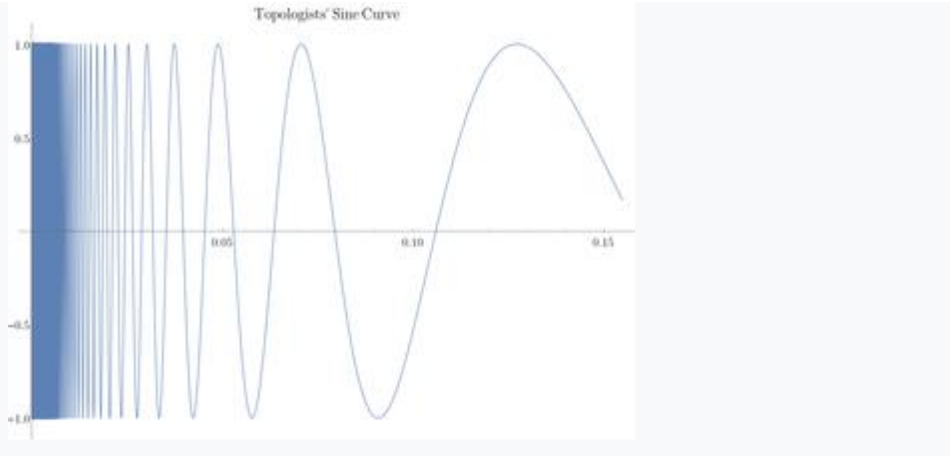
11.5.2 Arc Connectedness

A space *X* is said to be **arc-connected** or **arcwise connected** if any two distinct points can be joined by an *arc*, that is a path *f* which is a homeomorphism between the unit interval [0, 1] and its image $f([0, 1])$. It can be shown any Hausdorff space which is path-connected is also arc-connected. An example of a space which is path-connected but not arc-connected is provided by adding a second copy 0' of 0 to the nonnegative real numbers $[0, \infty)$. One endows this set with a partial order by specifying that $0' < a$ for any positive number a, but leaving 0 and 0' incomparable. One then endows this set with the *order topology*, that is, one takes the open intervals $(a, b) = \{x \mid a < x < b\}$ and the half-open intervals $[0, a) = \{x \mid 0 \leq x < a\}$, $[0', a) = \{x \mid 0' \leq x < a\}$ as a base for the topology. The resulting space is a T_1 space but not a Hausdorff space. Clearly 0 and 0' can be connected by a path but not by an arc in this space.

11.5.3 Local Connectedness

A topological space is said to be locally connected **at a point** x if every neighbourhood of x contains a connected open neighbourhood. It is **locally connected** if it has a base of connected sets. It can be shown that a space X is locally connected if and only if every component of every open set of X is open.

Similarly, a topological space is said to be **locally path-connected** if it has a base of path-connected sets. An open subset of a locally path-connected space is connected if and only if it is path-connected. This generalizes the earlier statement about $\mathbf{R}^n$ and $\mathbf{C}^n$, each of which is locally path-connected. More generally, any topological manifold is locally path-connected.



The topologist's sine curve is connected, but it is not locally connected

Locally connected does not imply connected, nor does locally path-connected imply path connected. A simple example of a locally connected (and locally path-connected) space that is not connected (or path-connected) is the union of two separated intervals in , such as R.

A classical example of a connected space that is not locally connected is the so called topologist's sine curve, defined as , with the Euclidean topology induced by inclusion in C.

THEOREMS

- **Main theorem of connectedness**: Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. If X is (path-)connected then the image $f(X)$ is (path-)connected. This result can be considered a generalization of the intermediate value theorem.
- Every path-connected space is connected.
- Every locally path-connected space is locally connected.
- A locally path-connected space is path-connected if and only if it is connected.
- The closure of a connected subset is connected. Furthermore, any subset between a connected subset and its closure is connected.
- The connected components are always closed (but in general not open)
- The connected components of a locally connected space are also open.
- The connected components of a space are disjoint unions of the path-connected components (which in general are neither open nor closed).
- Every quotient of a connected (resp. locally connected, path-connected, locally path-connected) space is connected (resp. locally connected, path-connected, locally path-connected).
- Every product of a family of connected (resp. path-connected) spaces is connected (resp. path-connected).

- Every open subset of a locally connected (resp. locally path-connected) space is locally connected (resp. locally path-connected).
- Every manifold is locally path-connected.

11.5.4 Graphs

Graphs have path connected subsets, namely those subsets for which every pair of points has a path of edges joining them. But it is not always possible to find a topology on the set of points which induces the same connected sets. The 5-cycle graph (and any n-cycle with $n > 3$ odd) is one such example.

As a consequence, a notion of connectedness can be formulated independently of the topology on a space. To wit, there is a category of connective spaces consisting of sets with collections of connected subsets satisfying connectivity axioms; their morphisms are those functions which map connected sets to connected sets (Muscat & Buhagiar 2006). Topological spaces and graphs are special cases of connective spaces; indeed, the finite connective spaces are precisely the finite graphs.

However, every graph can be canonically made into a topological space, by treating vertices as points and edges as copies of the unit interval (see topological graph theory#Graphs as topological spaces). Then one can show that the graph is connected (in the graph theoretical sense) if and only if it is connected as a topological space.

11.5.5 Stronger Forms of Connectedness

There are stronger forms of connectedness for topological spaces, for instance:

- If there exist no two disjoint non-empty open sets in a topological space, X, X must be connected, and thus hyperconnected spaces are also connected.
- Since a simply connected space is, by definition, also required to be path connected, any simply connected space is also connected. Note

however, that if the "path connectedness" requirement is dropped from the definition of simple connectivity, a simply connected space does not need to be connected.

- Yet stronger versions of connectivity include the notion of a contractible space. Every contractible space is path connected and thus also connected.

In general, note that any path connected space must be connected but there exist connected spaces that are not path connected. The deleted comb space furnishes such an example, as does the above-mentioned topologist's sine curve.

11.6 PATH-CONNECTED SPACES

Definition. *A* ***path*** *on a topological space X is a continuous map* $f : [0, 1] \to X$. *The path is said to* ***connect*** *x and y in X if f(0)=x and f(1)=y. X is said to be* ***path-connected*** *if any two points can be connected by a path.*

In a sense, path-connectedness is more active since one requires an explicit path to establish it, while the earlier connectedness is more passive since it simply indicates a failure to decompose as a topological disjoint union. The two are obviously related, starting from:

A topological space in which any two points can be joined by a continuous image of a simple arc; that is, a space X for any two points x_0 and x_1 of which there is a continuous mapping $f : I \to X$ of the unit interval $I = [0, 1]$ such that $f(0) = x_0$ and $f(1) = x_1$. A path-connected Hausdorff space is a Hausdorff space in which any two points can be joined by a simple arc, or (what amounts to the same thing) a Hausdorff space into which any mapping of a zero-dimensional sphere is homotopic to a constant mapping. Every path-connected space is connected (cf. Connected space). A continuous image of a path-connected space is path-connected.

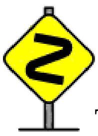
Path-connected spaces play an important role in homotopic topology. If a space X is path-connected and $x_0, x_1 \in X$, then the homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic, and this isomorphism is uniquely determined up to the action of the group $\pi_1(X, x_0)$. If $p: E \to B$ is a fibration with path-connected base B, then any two fibres have the same homotopy type. If $p: E \to B$ is a weak fibration (a Serre fibration) over a path-connected base B, then any two fibres have the same weak homotopy type.

The multi-dimensional generalization of path connectedness is k-connectedness (connectedness in dimension k). A space X is said to be connected in dimension k if any mapping of an r-dimensional sphere S^r into X, where $r \le k$, is homotopic to a constant mapping.

Theorem 1. *A path-connected space X is connected.*

Proof.

Suppose X is path-connected but not connected. There's a surjective continuous map $f : X \to \{0, 1\}$. Pick $x, y \in X$ such that $f(x)=0$ and $f(y)=1$. There's a path $g : [0, 1] \to X$ such that $g(0)=x$ and $g(1)=y$. Now the composition $f \circ g : [0,1] \to \{0, 1\}$ is continuous and surjective, which contradicts the fact that [0, 1] is connected. ♦



The converse is not true: *the topologist's sine curve is connected but not path-connected*.

Let $X = Y \cup Z \subset \mathbf{R}^2$, where $Y = \{0\} \times [0, 1]$, $Z = \{(x, \sin(1/x)) : 0 < x \le 1\}$.

To see why it's not path-connected, suppose $f : [0, 1] \to X$ is continuous and $f(0) = (0, 0)$, $f(1) = (1, \sin(1))$.

Let $\pi_x, \pi_y : X \to \mathbf{R}$ be projection maps to the x– and y-coordinates respectively. Then $\pi_x \circ f : [0, 1] \to \mathbf{R}$ contains 0 and 1, so its image is the whole [0, 1] by the intermediate value theorem. Hence, $\text{im}(f) \supset Z$. Pick points $t_0, t_1, t_2, \ldots \in [0, 1]$ such that $f(t_n) = ((2n\pi + \frac{\pi}{2})^{-1}, 1)$.

Notes

Since [0, 1] is compact, f is uniformly continuous. So there exists ε>0 such that whenever $t, u \in [0,1]$ satisfy $|t-u| < \epsilon,$ we have $|\pi_y(f(t)) - \pi_y(f(u))| < 1.$ Since [0, 1] is compact, we'll pick $m<n$ such that $|t_m - t_n| < \epsilon.$ By intermediate value theorem, there's a point u between t_m and t_n such that $\pi_x(f(u)) = ((2m\pi + \frac{3\pi}{2})^{-1}, -1).$ Then $|t_m - u| < \epsilon$ but $|\pi_y(f(t_m)) - \pi_y(f(u))| = |1-(-1)| = 2 > 1$ which is a contradiction.

[Notice it took quite a bit of effort to prove a seemingly obvious claim, and we needed compactness to prove it.]

Note also that X is closed in $\mathbf{R}^2$, and every point in Y is a point of accumulation of Z, so cl(Z) = X. In short, we have the first bummer.

Conclusion. *The closure of a path-connected subset Y of X is not necessarily path-connected.*

Proposition 2. *If* $f : X \to Y$ *is a continuous map of topological spaces and X is path-connected, then so is f(X).*

Proof.

For any $y_0, y_1 \in f(X)$ we can pick $x_0, x_1 \in X$ such that $f(x_0) = y_0, f(x_1) = y_1.$ Pick a path $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1.$ Then the composition gives a path $g \circ f : [0,1] \to Y$ which connects y_0 to $y_1.$ ♦

Proposition 3. *If* $\{Y_i\}$ *is a collection of path-connected subspaces of X and* $\cap_i Y_i \neq \emptyset,$ *then so is* $Y := \cup_i Y_i.$

Proof (Sketchy).

Pick $x \in \cap_i Y_i.$ If $y, z \in Y,$ then $y \in Y_i$ and $z \in Y_j$ for some indices i and j. Since Y_i and Y_j are path-connected and contain x, there's a path in Y_i connecting x to y and a path in Y_j connecting x to z. Hence, concatenating gives a path connecting y to z. ♦

Proposition 4. *If* $\{X_i\}$ *is a collection of path-connected spaces, then* $X := \prod_i X_i$ *is also path-connected.*

Proof.

Let $(x_i), (y_i) \in X$. Since each X_i is path connected, pick a path $f_i : [0,1] \to X_i$ such that $f_i(0) = x_i, f_i(1) = y_i$. Now let $f : [0,1] \to X$ be the path $f(t) := (f_i(t))_i \in X$.

To check that *f* is continuous, let's use the universal property of products. It suffices to show $\pi_i \circ f : [0,1] \to X_i$ is continuous for each *i*, where $\pi_i : X \to X_i$ is the projection map. But $\pi_i \circ f = f_i$, so we're done. ♦

11.6.1 Path-Connected Components

As before, we obtain the concept of path-connected components. We define, for points *x*, *y* in *X*, a relation *x* ~ *y* if and only if they belong to some path-connected component. Proposition 3 then tells us this gives an equivalence relation.

Definition. *The equivalence classes of the above-mentioned relation are called the* ***path-connected components****.*

Since a path-connected component is automatically connected, *each connected component is a disjoint union of path-connected components*.

11.6.2 Connected vs. path connected

A topological space X is said to be *connected* if it cannot be represented as the union of two disjoint, nonempty, open sets. While this definition is rather elegant and general, if X is connected, it does not imply that a path exists between any pair of points in X thanks to crazy examples like the *topologist's sine curve*:

$$X = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \quad \text{or} \quad y = \sin(1/x)\}.$$

Consider plotting X. The $\sin(1/x)$ part creates oscillations near the y-axis in which the frequency tends to ginfinity. After union is taken with the y-axis, this space is connected, but there is no path that reaches the y-axis from the sine curve.

How can we avoid such problems? The standard way to fix this is to use the path definition directly in the definition of connectedness. A topological space X is said to be *path connected* if for all $x_1, x_2 \in X$, there exists a path τ such that $\tau(0) = x_1$ and $\tau(1) = x_2$. It can be shown that if X is path connected, then it is also connected in the sense defined previously.

Another way to fix it is to make restrictions on the kinds of topological spaces that will be considered. This approach will be taken here by assuming that all topological spaces are manifolds. In this case, no strange things like (4.8) can happen,[4.7] and the definitions of connected and path connected coincide [451]. Therefore, we will just say a space is *connected*. However, it is important to remember that this definition of connected is sometimes inadequate, and one should really say that X is *path connected*.

11.7 SUMMARY

We study connected and path-connected space. We study Arc Connectedness and its examples. We study graph and its properties. We study Long line.

11.8 KEYWORD

Arc Connectedness : In topology and related branches of mathematics, a **connected** space is a topological space that cannot be represented as the union of two or more disjoint non-empty open subsets. **Connectedness** is one of the principal topological properties that are used to distinguish topological spaces

Graph : A diagram showing the relation between variable quantities, typically of two variables, each measured along one of a pair of axes at right angle

11.9 QUESTIONS FOR REVIEW

1. It's easy to see that any interval (closed, open or half-open) is path-connected. In particular, it's connected.
2. Hence $X = [0, 1) \cup (2, 3]$ is a disjoint union of $[0, 1)$ and $(2, 3]$, each of which is a path-connected component.
3. The squares [0, 1] × [0, 1] and (0, 1) × (0, 1) are path-connected by proposition 4.
4. Consider **Q** as a subspace of **R**. Since the connected components are singleton sets, the path-connected components can't break them down any further.
5. Take the topologist's sine curve $X = Y \cup Z$ above. *Y* and *Z* are both path-connected since they're homeomorphic to intervals. Since *X* is not path-connected, the path-connected components must be *Y* and *Z*. Note that *Y* is open while *Z* is closed in *X*. *This is one example where connected components decompose further into path-connected components.*

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11.11 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1

Q 2 Check in Section 1.3

Check in Progress-II

Answer Q. 1 Check in Section 4.1

Q 2 Check in Section 4.2

UNIT 12: COMPACT SPACE

STRUCTURE

12.0 OBJECTIVE

After going through this unit, you will be able to:

- Understand Local Connectedness and Compact Spaces
- Analyse Compactness and Nets
- Define Paracompact Spaces

12.1 INTRODUCTION

In mathematics, and more specifically in general topology, compactness is a property that generalizes the notion of a subset of Euclidean space being closed (that is, containing all its limit points) and bounded (that is, having all its points lie within some fixed distance of each other). Examples include a closed interval, a rectangle, or a finite set of points. This notion is defined for more general topological spaces than Euclidean space in various ways.

One such generalization is that a topological space is *sequentially* compact if every infinite sequence of points sampled from the space has an infinite subsequence that converges to some point of the space. The Bolzano–Weierstrass theorem states that a subset of Euclidean space is compact in this sequential sense if and only if it is closed and bounded. Thus, if one chooses an infinite number of points in the *closed* unit interval [0, 1] some of those points will get arbitrarily close to some real number in that space. For instance, some of the numbers 1/2, 4/5, 1/3, 5/6, 1/4, 6/7, … accumulate to 0 (others accumulate to 1). The same set of points would not accumulate to any point of the *open* unit interval (0, 1); so the open unit interval is not compact. Euclidean space itself is not compact since it is not bounded. In particular, the sequence of points 0, 1, 2, 3, … has no subsequence that converges to any real number.

Definition. *A collection* A *of subsets of X is said to be cover X or to be a covering of X if the union of elements of* A *is equal to X.*

Definition. *A collection* A *of open subsets of X is said to be a open covering of X if its union of elements of* A *is equal to X.*

Definition. *A space X is said to be compact if every open covering* A *of X contains a subcollection that also covers X.*

Example *The real line R is not connected.*

Let A = {(n, n + 2)/n * Z} *be a collection of open subsets of R whose union is*

R. But this collection does not have a finite subcollection that covers R.

Example Any compact subset of a Hausdorff space is closed.

Example Consider the upper-case letters of the alphabet { A, B, C, D, ... , Z } as being made up of (infinitely thin) lines. Classify them up to topological equivalence.

If we treat them as being made of lines of finite thickness (so that they are two-dimensional sets) how does the classification change?

If we treat them as being carved out of (say) wood (so that they are three-dimensional sets) does the classification change again?

Proof

Suppose $C \subset X$ is compact. To show that $X - C$ is open we take $x \in X - C$ and try and show that x is in an open subset of $X - C$.

For each $y \in C$ we can find disjoint open sets U_y and V_y separating x and y: $x \in U_y$ $y \in V_y$. The set $\cap U_y$ where the intersection is over all $y \in C$ does not meet C and hence is in $X - C$. Unfortunately, it is not necessarily open since a topology $\mathcal{T}$ is not closed under infinite intersections. However, since C is compact, we may discard all but finitely many of the V_y's and the intersection of the corresponding U_y's *will* be the open set we need.

The answers will depend on the way you write!

For this particular sans-serif font A ≅ R,

C ≅ G ≅ I ≅ J ≅ L ≅ M ≅ N ≅ S ≅ U ≅ V ≅ W ≅ Z,

D ≅ O, E ≅ F ≅ T ≅ Y, H ≅ K and the rest (B, P, Q, X) are distinct.

Regarding the letters as having finite thickness gives some more homeomorphisms. For example **P** ≅ **O**.

So one gets **A** ≅ **D** ≅ **O** ≅ **P** ≅ **Q** ≅ **R**, **B** on its own and all the rest homeomorphic to one another.

Note that the equivalence classes are distinguished by the "number of holes".

Making the letters 3-dimensional does not produce any further homeomorphisms.

12.2 COMPACT SPACE

Definition. *If Y is the subspace of X, a collection* A *of subset of X is said to cover Y if the union of this element contains Y.*

A topological space is compact if every open cover of X has a finite subcover. In other words, if X is the union of a family of open sets, there is a finite subfamily whose union is X. A subset A of a topological space X is compact if it is compact as a topological space with the relative topology (i.e., every family of open sets of X whose union contains A has a finite subfamily whose union contains AThen {Aα1 ,Aα2 , . . . ,Aαn} is the finite subcollection of A that covers Y.

Corollary

Any real-valued function on a closed bounded interval is bounded and attains its bounds.

Proof

The closed bounded interval is compact and hence its image is compact and hence is also a closed bounded subset which is in fact an interval also, by connectedness. Thus the function is bounded and its image is an interval [*p*, *q*]. It attains its bounds at points mapped to *p* and *q*.

Definitions

A topological space is compact if every open covering has a finite sub-covering.

An open covering of a space X is a collection $\{U_i\}$ of open sets with $\bigcup_{i \in I} U_i = X$ and this has a finite sub-covering if a finite number of the U_i's can be chosen which still cover X.

The most important thing is what this means for R with its usual metric.

Theorem

The interval [0, 1] *is compact under the usual metric on* R.

Proof

Let $\{U_i\}$ be an open covering of [0, 1]. The trick is to consider the set $A = \{x \in [0, 1] \mid [0, x]$ can be covered by finitely many of the U_i's$\}$.

Then use the Completeness property of R to take α to be the least upper bound of A.

Suppose $\alpha < 1$. Then α is contained in some open set U_{i0} and so lies in an ε-neighbourhood lying in U_{i0} .

But now $[0, \alpha - \varepsilon/2]$ is covered by finitely many of the U_i's and so this collection, together with U_{i0} covers $[0, \alpha + \varepsilon/2]$ which contradicts the definition of α.

□

A similar proof shows that *any* closed bounded interval of R is compact. We will see later that in fact any closed bounded *subset* of R (with its usual metric) is compact.

Theorem

A compact subset of R *with its usual metric is closed and bounded.*

Proof

If a set $A \subset$ R is not closed then there is a limit point $p \notin A$. Then cover A by complements of *closed* ε-neighbourhoods of p for p = $1, {}^1/_2 , {}^1/_3 , \ldots$.

For example If $A = (0, 1)$ and $p = 0$ then $(0, 1) =$ $({}^1/_2 ,1) \cup ({}^1/_3 ,1) \cup ({}^1/_4 ,1) \cup \ldots$

We cannot take a finite subcover to cover A.

A similar proof shows that an *unbounded* set is not compact.

Properties of compactness

1. *Continuous images of compact sets are compact.*

   That is , if $f : C \to Y$ is continuous and C is compact then $f(C)$ is compact also.

   Proof

   Let $\{U_i\}$ be an open cover of $f(C)$. Then $\{f^{-1}(U_i)\}$ is an open cover of C and can therefore be reduced to a finite subcover. The corresponding collection of U_i's will be a finite sub-cover of $f(C)$.

   Corollary

   If X is compact and ~ is any equivalence relation then X/~ is compact.

   Proof

   The natural map $p: X \to X/\sim$ is continuous and onto.

2. *Any closed subset of a compact space is compact.*

   Proof

   If $\{U_i\}$ is an open cover of $A \subset C$ then each $U_i = V_i \cap A$ with V_i oopen in C. Then the collection $\{V_i\}$ together with the open set C - A cover C and hence have a finite subcover. The corresponding U_i's then cover A.

Therefore, f^{-1} is continuous.

Therefore, f is a homeomorphism. 2

Example Any closed subset of a compact space is compact.

Proof

If $\{U_i\}$ is an open cover of $A \subset C$ then each $U_i = V_i \cap A$ with V_i oopen in C. Then the collection $\{V_i\}$ together with the open set C - A cover C and hence have a finite subcover. The corresponding U_i's then cover A.

SEQUENTIAL COMPACTNESS

For metric spaces there is another, perhaps more natural way of thinking about compactness. It is based on the following classical result.

The Bolzano-Weierstrass theorem

Every bounded sequence in **R** *has a convegent subsequence.*

This is attributed to the Czech mathematician Bernhard Bolzano (1781 to 1848) and the German mathematician Karl Weierstrass (1815 to 1897).

From this we are led to the generalisation:

Definition

A metric space is sequentially compact if every bounded infinite set has a limit point.

The main result is:

Theorem

A compact metric space is sequentially compact.

Proof

Let *A* be an infinite set in a compact metric space *X*. To prove that *A* has a limit point we must find a point *p* for whicch every open neighbourhood of *p* contains infinitely many points of *A*.

Suppose that no such point existed. Then every point of *X* has an open neighbourhood containing only finitely many points of *A*. These sets form an open cover of *X* and extracting a finite open cover gives a covering of *X* meeting *A* in only finitely many points. This is impossible since $A \subset X$ and *A* is infinite.

Corollary

In a compact metric space every bounded sequence has a convergent sub-sequence.

Proof

Given the above limit point *p*, take $x_{i1}$to be in a 1-neighbourhood of *p*, $x_{i2}$to be in a $^1/_2$-neighbourhood of *p*, ... and we get a sub-sequence converging to *p*.

Together with the Heine-Borel theorem this implies the Bolzano-Weierstrass theorem.

Remark

In fact, a metric space is compact if and only if it is sequentially compact. The proof that sequentially compact $\Rightarrow$ comact is harder.

Some Important Summary

A *topological space* is a set *X* together with a set $\mathcal{T}$ of subsets called "open sets" such that:

the subsets $\varnothing$ and X $\in$ $\mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions and finite intersections.

Notes

Closed sets are complements of open sets.

A *basis* for a topology $\mathcal{T}$ is a set $\mathcal{B}$ of subsets such that any set in $\mathcal{T}$ can be written as a union of sets in $\mathcal{B}$. In a metric space, the ε-neighbourhoods form a basis for the topology.

Some examples of topological spaces are:

Any metric space with the open sets defined as above,

The trivial topology on any set X: = { ∅, X },

Certain topologies on finite sets. e.g. the Sierpinski topology:

X = {a, b}, = { ∅, {a}, {a, b}},

The *cofinite* (or Zariski) topology in which proper *closed* sets are the finite sets,

The *co-countable* topology in which proper *closed* sets are the countable sets.

The *interior int*(A) of a set A in a topological space is the largest open subset of A.

The *closure cl*(A) of a subset A is the smallest closed subset containing A.

A function $f : X \to Y$ between topological spaces is *continuous* if $f^{-1}(A)$ is open in X whenever A is open in Y.

A continuous bijection whose inverse function is also continuous is called a *homeomorphism* or *topological isomorphism*.

Example. Recall the definitions:

A *closed interval* is the set $[a, b] = \{x \in \mathbf{R} \mid a \le x \le b \}$

An *open interval* is the set $(a, b) = \{x \in \mathbf{R} \mid a < x < b \}$ and there are also open intervals

$(a, \infty) = \{x \in \mathbf{R} \mid a < x \}$, $(-\infty, b) = \{x \in \mathbf{R} \mid x < b \}$ and $(-\infty, \infty) = \mathbf{R}$.

A *half-open interval* is the set $[a, b) = \{x \in \mathbf{R} \mid a \le x < b \}$ or $(a, b] = \{x \in \mathbf{R} \mid a < x \le b \}$ and there are also half-open intervals $[a, \infty)$ and $(-\infty, b]$ defined similarly.

Draw the graphs of continuous maps which show that any two closed intervals are homeomorphic (topologically equivalent).

Prove that any two *finite* open intervals are homeomorphic.

Prove that the open interval (-p/2, p/2) is homeomorphic to the real line **R**.

[Hint: Consider the map $f(x) = \tan(x)$.]

Prove that any two open intervals are homeomorphic.

Prove that any two half-open intervals are homeomorphic.

Solution To map the interval $[a, b]$ to $[c, d]$ take the linear map whose graph is the straight line joining the point (a, c) to (b, d).

The same map works for the finite open intervals also.

The map $x \mapsto \tan(x)$ maps the finite open interval (-p/2, p/2) to the whole line **R** in a bijective way with the continuous inverse $y \mapsto \tan^{-1}(y)$.

The same map shows that any open interval of the form (a, ∞) or $(-\infty, b)$ is homeomorphic to a subinterval of (-p/2, p/2).

Hence any open intervals are homeomorphic to finite open intervals and hence to each other.

Similar methods show that all half-open intervals are homeomorphic to one another.

Check In Progress

Q 1. Define Compact Space.

Solution

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Q 2. A compact subset of R with its usual metric is closed and bounded.

A compact subset of R with its usual metric is closed and bounded.

Solution

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Notes

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12.3 THE LINDEL¨OF PROPERTY

In this section, we will recollect some main definitions, will summarize the main results about Lindel¨of spaces, and present in depth several examples about the interrelation of some main topological notions in Lindel¨of spaces. We will also aim at providing detailed visualisation of those examples that will enable the reader to better grasp the core ideas.

A topological space is said to be Lindel¨of, or have the Lindel¨of property, if every open cover of X has a countable subcover. The Lindel¨of property was introduced by Alexandroff and Urysohn in 1929, the term 'Lindel¨of' referring back to Lindel¨of's result that any family of open subsets of Euclidean space has a countable sub-family with the same union. Clearly, a space is compact if and only if it is both Lindel¨of and countably compact, though weaker properties, for example pseudocompactness, imply compactness in the presence of the Lindel¨of property. The real line is a Lindel¨of space that is not compact and the space of all countable ordinals ω1 with the order topology is a countably compact space that is not Lindel¨of. It should be noted that some authors require the Hausdorff or regular (which we take to include T1) separation axioms as part of the definition of many open covering properties (c.f. [E]). For any unreferenced results in this article we refer the reader to [E].

There are a number of equivalent formulations of the Lindel¨of property: (a) the space X is Lindel¨of; (b) X is [ω1,∞]-compact (see the article by Vaughan in this volume); (c) every open cover has a countable refinement; (d) every family of closed subspaces with the countable intersection property1 has non-empty intersection; (e) (for regular spaces) every open cover of X has a countable subcover V such that {V : V ∈ V} covers X (where A denotes the closure of A in X). In the class of locally compact spaces, a space is Lindel¨of if and only if it is σ-compact (i.e., is a countable union of compact spaces) if and only if it can be written as an increasing union of countably many open sets each of which has compact closure.

It is an important result that regular Lindel¨of spaces are paracompact, from which it follows that they are (collectionwise) normal. Conversely, every paracompact space with a dense Lindel¨of subspace is Lindel¨of (in particular, every separable paracompact space is Lindel¨of) and every locally compact, paracompact space is a disjoint sum of clopen Lindel¨of subspaces. A related result is that any locally finite family of subsets of a Lindel¨of space is countable.

Closed subspaces and countable unions of Lindel¨of spaces are Lindel¨of. Continuous images of Lindel¨of spaces are Lindel¨of and inverse images of Lindel¨of spaces under perfect mappings, or even closed mappings with Lindel¨of fibres, are again Lindel¨of. In general, the Lindel¨of property is badly behaved on taking either (Tychonoff) products or inverse limits.

The Tychonoff product of two Lindel¨of spaces need not be Lindel¨of or even normal, although any product of a Lindel¨of space and a compact space is Lindel¨of and countable products of Lindel¨of scattered spaces are Lindel¨of [HvM, Chapter 18, Theorem 9.33]. It is also true that both the class of Cech com- ˇ plete Lindel¨of and Lindel¨of Σ-spaces are closed under countable products. The Sorgenfrey line, which one obtains from the real line by declaring every interval of the form (a, b] to be open, is a simple example of a Lindel¨of space with non-normal square. Even more pathological examples are possible: Michael constructs a Lindel¨of space, similar to the Michael line, which has non-normal product with a subset of the real line and, assuming the Continuum Hypothesis,

constructs a Lindel¨of space whose product with the irrationals is non-normal. Details and further results may be found in [KV, Chapter 18] and Section 9 of [HvM, Chapter 18]. A space is said to be realcompact if it is homeomorphic to a closed subspace of the Tychonoff product R^{κ} for some κ. Every regular Lindel¨of space is realcompact and, whilst the inverse limit of a sequence of Lindel¨of spaces need not be normal, both inverse limits and arbitrary products of realcompact spaces are realcompact. Hence arbitrary products and inverse limits of regular Lindel¨of spaces are realcompact. In fact a space is realcompact if and only if it is the inverse limit of a family of regular Lindel¨of spaces.

Second countable spaces (i.e., spaces with a countable base to the topology) are both Lindel¨of and separable. The Sorgenfrey line is an example of a separable, Lindel¨of space that is not second countable. On the other hand, if X is metrizable (or even pseudometrizable), then X is second countable if and only if it is separable if and only if it has the countable chain condition if and only if it is Lindel¨of. By Urysohn's Metrization Theorem, a space is second countable and regular if and only if it is a Lindel¨of metrizable space if and only if it can be embedded as subspace of the Hilbert cube.

A space X is said to be hereditarily Lindel¨of if every subspace of X is Lindel¨of. Since any space can be embedded as a dense subspace of a (not necessarily Hausdorff) compact space, not every Lindel¨of space is hereditarily Lindel¨of. However, a space is hereditarily Lindel¨of if and only if every open subspace is Lindel¨of if and only if every uncountable subspace Y of X contains a point y whose every neighbourhood contains uncountably many points of Y . A regular Lindel¨of space is hereditarily Lindel¨of if and only of it is perfect and hereditarily Lindel¨of spaces have the countable chain condition but need not be separable.

In fact, for regular spaces there is a complex and subtle relationship between the hereditary Lindel¨of property and hereditary separability2 (both of which follow from second countability). An hereditarily Lindel¨of regular space that is not (hereditarily) separable is called an L-space; an hereditarily separable regular space that is not (hereditarily) Lindel¨of is called an S-space. The existence of S- and L-spaces is, to a certain extent, dual and depends strongly on the model of set theory. For example, the existence of a Souslin line implies the existence of both S- and L-spaces, MA + ¬CH is consistent with the existence of S- and L-spaces but implies that neither compact S- nor compact L-spaces exist. However, the duality is not total: Todorˇcevi´c [11] has shown that it is consistent with MA that there are no S-spaces but that there exists an L-space, i.e., that every regular hereditarily separable space is hereditarily Linde¨of but that there is a non-separable, hereditarily Lindel¨of regular space. It is currently an open question whether it is consistent that there are no L-spaces. For further details about S and L see Roitman's article [KV, Chapter 7], or indeed [11]. It is fair to say that the S/L pathology, along with Souslin's Hypothesis and the Normal Moore Space Conjecture, has been one of the key motivating questions of set-theoretic topology and it crops up frequently in relation to other problems in general topology, such as: the metrizability of perfectly normal manifolds [10]; Ostaszewski's construction of a countably compact, perfectly normal noncompact space [9]; and the existence of a counter-example to Katˇetov's problem 'if X is compact and X2 is hereditarily normal, is X metrizable?' [5].

The Lindel¨of degree or number, L(X), of a space X is the smallest infinite cardinal κ for which every open cover has a subcover of cardinality at most κ. The hereditary Lindel¨of degree, hL(X), of X is the supremum of the cardinals L(Y) ranging over subspaces Y of X. The Lindel¨of degree of a space is one of a number of cardinal invariants or cardinal functions one might assign to a space. Cardinal functions are discussed in the article by Tamano in this volume, however, one result due to Arkhangel0 skiˇı [1] is worth particular mention here. The character χ(x, X) of a point x in the space X is smallest cardinality of a local base at x and the character χ(X) of the space X is the supremum sup{χ(x, X) : x ∈ X}. A space with countable character is said to be first-countable. Arkhangel0 skiˇı's result says that the cardinality of a Hausdorff space X is at most 2L(X).χ(X) . In the countable case this theorem tells us that the cardinality of a first-countable, Lindel¨of Hausdorff space is at most the continuum, 2ℵ0 , and that, in particular, the cardinality of a firstcountable, compact Hausdorff space is at most the continuum.3 This impressive result solved a problem posed thirty years earlier by Alexandroff and Urysohn (whether a first-countable compact space could have cardinality greater than that of the continuum), but was, moreover, a model for many other results in the field. The theorem does not remain true if we weaken first-countability, since it is consistent that the cardinality of a regular, (zero-dimensional even) Lindel¨of Hausdorff space with countable pseudo-character can be greater than that of the continuum [12], and Lindel¨of spaces can have arbitrary cardinality. However, de Groot has shown that the cardinality of a Hausdorff space X is at most 2hL(X) [KV, Chapter 1, Cor. 4.10]. For a much more modern proof of Arkhangel0 ski˘ı's theorem than the ones given in [1] or [KV, Chapter 1], we refer the reader to Theorem 4.1.8 of the article by Watson in [HvM].

A space is compact if and only if every infinite subset has a complete accumulation point if and only if every increasing open cover has a finite subcover and a space is countably compact if and only if every countably infinite subset has a complete accumulation point. However, the requirement that every uncountable subset has a complete accumulation point is implied by, but does not characterize the Lindel¨of property. Spaces satisfying this property are called linearly Lindel¨of, since they turn out to be precisely those spaces in which every open cover that is linearly ordered by inclusion has a countable subcover. Surprisingly little is known about such spaces. There are (somewhat complex) examples of regular linearly Lindel¨of, non-Lindel¨of spaces in ZFC, but there is, at present, no known example of a normal linearly Lindel¨of, non-Lindel¨of spaces under any set theory. Such a space would be highly pathological: the problem intrinsically involves singular cardinals and any example is a Dowker space, that is, a normal space which has non-normal product with the closed unit interval [0, 1]. Nevertheless one can prove some interesting results about linearly Lindel¨of spaces, for example every first-countable, linearly Lindel¨of Tychonoff space has cardinality at most that of the continuum, generalizing the theorem of Arhangel'ski's result mentioned above. For more on linearly Lindel¨of spaces see the paper by Arkhangel0 ski˘ı and Buzyakova [2].

One important sub-class of Lindel¨of spaces, the Lindel¨of Σ spaces, deserves mention. The notion of a Σ-space was introduced by Nagami [8], primarily to provide a class of space in which covering properties behave well on taking products. It turns out that there are a number of characterizations of Lindel¨of Σ-spaces, two of which we mention here.

A Tychonoff space is Lindel¨of Σ if it is the continuous image of the pre-image of a separable metric space under a perfect map. An equivalent (categorical) definition is that the class of Lindel¨of Σ-spaces is the smallest class containing all compact spaces and all separable metrizable spaces that is closed under countable products, closed subspaces and continuous images. So, as mentioned above, countable products of Lindel¨of Σspaces are Lindel¨of Σ. Every σ-compact space, and hence every locally compact Lindel¨of space, is a Lindel¨of Σ-space. Lindel¨of Σ-spaces play an important rˆole in the study of function spaces (with the topology of pointwise convergence). For details, see the article on Cp-theory by Arkhangel0 skiˇı [HvM, Chapter 1].

There are several strengthenings and weakenings of the Lindel¨of property in the literature for example: almost Lindel¨of, n-starLindel¨of, totally Lindel¨of, strongly Lindel¨of, Hurewicz, subbase Lindel¨of. We mention one in passing. A space is weakly Lindel¨of if any open cover has a countable subfamily V such that S {V : V ∈ V} is dense in X. Weakly Lindel¨of spaces are of some interest in Banach space theory [HvM, Chapter 16] and, assuming CH, the weakly Lindel¨of subspaces of βN are precisely those which are C ∗ -embedded into βN (1.5.3 of [KV, Chapter 11]). Covering properties such as para- or metaLindel¨of belong more properly to a discussion of generalizations of paracompactness.

Finally, we list a number of interesting results concerning the Axiom of Choice and the Lindel¨of property. The Countable Axiom of Choice is strictly stronger than either of the statements 'Lindel¨of metric spaces are second countable' or 'Lindel¨of metric spaces are separable' [7]. In Zermelo-Fraenkel set theory (without choice) the following conditions are equivalent: (a) N is Lindel¨of; (b) R is Lindel¨of; (c) every second countable space is Lindel¨of; (d) R is hereditarily separable; (e) f : R → R is continuous at x iff it is sequentially continuous at x; and (f) the axiom of countable choice holds for subsets of R [6]. There are models of ZF in which every Lindel¨of T1-space is compact [3] and models in which the space ω1 is Lindel¨of but not countably compact [4].

Theorem Let X be a metric space. Then the following are equivalent:

1. X is Lindel¨of,

2. X is hereditarily Lindel¨of,

3. X is second countable,

4. X is separable,

5. X satisfies the countable chain condition. The following example shows that in general, separability does not follow from being CCC and Lindel¨of.

Claim. X is Lindel¨of.

Proof. We will use the fact that X is Lindel¨of iff from any cover with basic open sets we can find a countable subcover. Let $\gamma = \{O(ak, bk, \alpha k) : k \in K, \alpha k < \omega 1\}$ be an open cover of X with basic open sets. Let us note that $\alpha = 0$ must be among those $\{\alpha k : \alpha k < \omega 1\}$, otherwise points from $X \cap (R \times \{0\})$ will not be covered. Let A = S $\alpha \in \omega 1$ $Q\alpha$; note that A is uncountable. Then $A \subset R$ and since R is metric with a countable base, so is A; hence, A is Lindel¨of. Then $U = \{(ak, bk) \cap A : k \in K\}$ is an open cover of A in the Euclidean topology and we can choose a countable subcover which we shall denote by $Y = \{(an, bn) \cap A : n \in N\}$. Then $\{\alpha n : n \in N\}$ is a countable set of countable ordinals, hence $\alpha 0 = \sup\{\alpha n : n \in N\}$ is at most countable. Let us show that $\{O(an, bn, \alpha n) : n \in N\}$ covers all but countably many elements of X.

Check In Progress

Q 1. Give Introduction of The Lindel¨of Property.

Solution

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Q 2. Let X be a metric space. Then the following are equivalent:

1. X is Lindel¨of,

Solution

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12.4 DIFFERENT FORMS OF COMPACTNESS AND THEIR RELATION

Given a topological space X one can define several notion of compactness:

X is **compact** if every open cover has a finite subcover.

X is **sequentially compact** if every sequence has a convergent subsequence.

X is **limit point compact** (or Bolzano-Weierstrass) if every infinite set has an accumulation point.

X is **countably compact** if every countable open cover has a finite subcover.

X is **σ-compact** if it is the union of countably many compact subspaces.

X is **pseudocompact** if its if its image under any continuous function to RR is bounded.

X is **paracompact** if every open cover admits an open locally finite refinement (i.e. every point of X has a neighborhood small enough to intersect only finitely many members of the cover).

X is **metacompact** if every open cover admits a point finite open refinement (i.e. if every point of X is in only finitely many members of the refinement).

X is **orthocompact** if every open cover has an interior preserving open refinement (i.e. given an open cover there is a open subcover such that at any point, the intersection of all open sets in the subcover containing that point is also open).

X is **mesocompact** if every open cover has a compact-finite open refinement (i.e. given any open cover, we can find an open refinement such that every compact set is contained in finitely many members of the refinement).

So, there are quite a few notions of compactness (there are surely more than those I quoted up here). The question is: where are these definitions systematically studied? What I'm interested in particular is knowing when does one imply the other, when does it not (examples), &c.

I can fully answer the question for the first three notions:

Compact and first-countable --> Sequentially compact.

Sequentially compact and second-countable --> Compact.

Sequentially compact --> Limit-point compact.

Limit point compact, first-countable and T1 --> Sequentially compact.

but I'm absolutely ignorant about the other cases. Has this been systematically studied somewhere? If so, where?

Example : Let N be the set of all natural numbers, and let τI is indiscrete topology on N. N} is b-open cover (respectively, gopen cover, bg-open cover) of N which has no finite subcover.∈Evidently, N is compact space. However, it is not b-compact (respectively, not gcompact, not bg-compact) space, since {{n}: n

Example Let H and X-H is finite} be a∈ X: 0⊆ {H∪ {0} with τ= Ᵽ(N)∪: Let X= N topological space, where Ᵽ(N) is the power set of the natural numbers. A.∈Then, any g-open cover of X must contains a g-open set A, such that 0 Implies, X-A is finite. Hence, X is g-compact space. But X is neither b-compact space nor bg-compact. Since, (E O∈{{n}: n∪ {0})∪+ + } is b-open cover, also bgopen cover, of X which has no finite subcover.

Proposition : A bg-closed subset of a bg-compact space is bg-compact relative to X. I} be a∈α: α

Proof: Let A be a bg-closed subset of a bg-compact space X. Let {G (X-A) is bg-open cover of X.∪I}∈α: αcover of A by bg-open sets in X. Then C= {G ∪1αSince X is bg-compact space, C is reducible to a finite subcover of X, and X= G n. Therefore A is bgcompact relative to X.α G∪… ∪2α G∪1α G⊆ (X-A). Hence, A∪nα G∪… ∪2αG Thus every bg-closed subset of a bg-compact space is bg-compact.

Y is said to be bg-continuous (respectively,→

Definition : [3] A function *f*: X bg-irresolute) if *f*-1 (V) is bg-closed in X for every closed (respectively, bg-closed) set V of Y. Y is said to be bg-continuous (respectively, bgirresolute) if *f*-1→

Proposition : A function *f*: X (V) is bg-open in X for every open (respectively, bg-open) set V of Y. P

Proposition: A bg-continuous image of a bg-compact space is compact. Y is a bg-continuous function from a bg-compact space X onto a→Proof: Let *f*: X I} be an open cover of Y. Then {*f*∈α: αspace Y. Let {A -1 I} is a bg-open∈α): α(A cover of X. Since X is bg-compact, it has a finite subcover say {*f*-1 i): i= 1, 2, …,α(A i: i= 1, 2, …, n} is a cover of Y, which is finite. Therefore Yαn}. Since *f* is onto, {A is compact

Lemma : The continuous and open mapping from a space X into a space Y is bg-irresolute. (Y, σ) be a continuous and open function, let V be a bg-open→Proof: Let *f*: (X, τ) subset of Y. Let F be a closed subset of *f*-1 (V) in X. Implies, *f*(F) is closed subset of *f*(*f*-1⊆Y, and *f*(F) σ-Intb(V), so *f*-1⊆(V))= V. Then *f*(F) *f*⊆(*f*(F)) -1 (σ-Intb(V)). τ-Intb(*f*-1⊆Therefore F (V)). Hence, *f* -1 (V) is bg-open subset of X. Thus *f* is bgirresolute.

Proposition : For a space X, the following statements are equivalent.

1. X is bg-compact.

2. Any family of bg-closed subsets of X satisfying the finite intersection property has a non-empty intersection.

Notes

3. Any family of bg-closed subsets of X with empty intersection has a finite subfamily with empty intersection.

Proof: 1 2: I} be a family of bg-closed subsets of X∈Let X be a bg-compact space and {Fi: i .ϕ I} ≠∈which satisfying the finite intersection property. To prove that ∩ {Fi: i I}∈ I}, so {X-Fi: i∈ {X-Fi: i∪. Then X= ϕ I} =∈Suppose the inverse, i.e., ∩ {Fi: i is bg-open cover of X which is bg-compact space. Implies, there exists a finite subset , which is a contradiction.ϕ I} =∈ I0} and so ∩ {Fi: i∈ {X-Fi: i∪I0 of I such that X= .ϕ I} ≠∈Thus, ∩ {Fi: i

2 1: Suppose 2 hold and X is not bg-compact space, then there exists a bg-open cover I} of X has no finite subcover. Thus, for any finite subset I0 of I, we have X≠∈{Vi: i I} satisfies∈. Therefore the family {(X-Vi): iϕ I0} ≠∈ I0}. So, ∩{(X-Vi): i∈ {Vi: i∪ the finite intersection property, then it has a non-empty intersection, i.e., ∩{(X-Vi): I}, which is a contradiction. Hence, X is bg-compact∈ {Vi: i∪. Implies, X≠ ϕ I} ≠∈i space.

2 3: Obvious. The notions of a filter and a net play an important role in all compact spaces. Therefore, we introduce the following notions which will be used in this paper to give some characterizations of bg-compact spaces in terms nets and filter bases.

Definition: Let A be a subset of a space X. A point x . The setϕpoint of A if for each bg-open set U containing x, we have (U- {x}) ∩ A ≠ of all bg-limit points of A is called a bg-derived set of A and denoted by Dbg(A). I} is bg-accumulates at a∈α: α

Definition: Let I be a directed set. A net φ ={x I, there∈0αpoint x of a space X if for each U bg-open set containing x and for each U.∈α0 such that xα ≥ αis some The net φ is bg-converges to a point x of X if for each U bg-open set containing x, 0.α ≥ α U for all ∈α I such that x∈0αthere is.

Definition 4 (Limit point Compact) K ⊆ X is limit point compact if every infinite subset of it has a limit point.

Recall that x ∈ X is a limit point of the set K if every neighbourhood of x intersects K − {x}. For a net {xλ}λ∈Λ we define x as its cluster point4 if for every neighbourhood O of x, the index set {λ : xλ ∈ O} is cofinal in Λ. It can be shown that x is a cluster point of the net {xλ}λ∈Λ iff some subnet {xλγ }γ∈Γ converges to x. Similarly x is a limit point of the set K iff there exists some net x 6∈ {xλ}λ∈Λ ⊆ K that converges to x. We will also need the notion of ω-limit point. A point x ∈ X is an ω-limit point of the set K if every neighbourhood of x intersects K at infinitely many points5 . Apparently ω-limit point is bona fide a limit point while the converse is true only in T1 space. Note that there is no need to define ω-cluster point (it coincides with cluster point).

Theorem 1 K ⊆ X is countably compact iff every infinite subset of it has an ω-limit point iff every sequence in it has a cluster point.

Proof: Suppose every sequence has a cluster point and let A be an infinite set, then we can find a sequence (with distinct elemets) in A whose cluster point clearly is an ω-limit point of A. On the other hand, suppose every infinite subset has an ω-limit point. Consider any sequence {xn}, let its distinct elements be {xnm}, which we assume is an infinite set (otherwise we are done). Apparently any ω-limit point of {xnm} is a cluster point of {xn}.

Suppose X is countably compact, let {xn} be a sequence in X . Denote Bn := cl({xi}∞ i=n), hence ∃x ∈ ∩∞ n=1Bn. Any neighbourhood O 3 x must intersect infinitely many elements of the sequence {xn} (again we omit the uninteresting case where the sequence only has finitely many distinct elements) since otherwise it would imply that ∩∞ n=1Bn = ∅. We have used the fact that for any open set O, O∩A = ∅ ⇐⇒ O∩cl(A) = ∅. On the other hand, suppose every sequence has a cluster point, and let Bn be a sequence of closed sets that satisfy the finite intersection property. We can choose xn ∈ B1 ∩ . . . ∩ Bn. Any cluster point of {xn}, say x, must belong to ∩∞ n=1Bn.

12.5 SUMMARY

We study in this unit compact space and its properties with examples. We study Countably in this unit. We study different types of compactness and its properties with examples.

12.6 KEYWORD

COMPACT : Exert force on (something) so that it becomes more dense; compress

COUNTABLY : Capable of being
counted: countable items; countable sins

ORTHOCOMPACT : ***Orthocompact***; this shows that orthocompactness is preserved by closed maps i

12.7 QUESTIONS FOR REVIEW

1 A closed sub-space of a Lindelof space is Lindelof.
2 Every second countable space is a Lindelof space.

3 Any uncountable discrete topological space is not Lindel¨of.

4. The space [0, ω1), with the order topology, is not Lindel¨of.

5. A space X is Lindel¨of if and only if every family of closed nonempty subsets of X which has the countable intersection property has a non-empty intersection.

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12.9 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1

Q 2 Check in Section 2

Check in Progress-II

Answer Q. 1 Check in Section 4

Q 2 Check in Section 4

UNIT 13: DIFFERENT KIND OF COMPACTNESS

STRUCTURE

13.0 OBJECTIVE

- After going through this unit, you will be able to:

Discuss compact set in the real line, sequentially and countably compact sets

- Describe Bolzano-Weierstrass property and sequential compactness,
- completeness and finite intersection property

Explain continuous functions and compact sets, characterisation of continuous functions

- Interpret limit point compactness and local compactness

13.1 INTRODUCTION

Local Compactness

In mathematics, and more specifically in general topology, **compactness** is a property that generalizes the notion of a subset of Euclidean space being closed (that is, containing all its limit points) and bounded (that is, having all its points lie within some fixed distance of each other). Examples include a closed interval, a rectangle, or a finite set of points. This notion is defined for more general topological spaces than Euclidean space in various ways.

One such generalization is that a topological space is *sequentially* compact if every infinite sequence of points sampled from the space has an infinite subsequence that converges to some point of the space. The Bolzano–Weierstrass theorem states that a subset of Euclidean space is compact in this sequential sense if and only if it is closed and bounded. Thus, if one chooses an infinite number of points in the *closed* unit interval [0, 1] some of those points will get arbitrarily close to some real number in that space. For instance, some of the numbers 1/2, 4/5, 1/3, 5/6, 1/4, 6/7, … accumulate to 0 (others accumulate to 1). The same set of points would not accumulate to any point of the *open* unit interval (0, 1); so the open unit interval is not compact. Euclidean space itself is not compact since it is not bounded. In particular, the sequence of points $0, 1, 2, 3, \ldots$ has no subsequence that converges to any real number.

Apart from closed and bounded subsets of Euclidean space, typical examples of compact spaces include spaces consisting not of geometrical points but of functions. The term *compact* was introduced into mathematics by Maurice Fréchet in 1904 as a distillation of this concept. Compactness in this more general situation plays an extremely important role in mathematical analysis, because many classical and important theorems of 19th-century analysis, such as the extreme value theorem, are easily generalized to this situation. A typical application is furnished by the Arzelà–Ascoli theorem or the Peano existence theorem, in which one is able to conclude the existence of a function with some required properties as a limiting case of some more elementary construction.

Various equivalent notions of compactness, including sequential compactness and limit point compactness, can be developed in general metric spaces. In general topological spaces, however, different notions of compactness are not necessarily equivalent. The most useful notion, which is the standard definition of the unqualified term *compactness*, is phrased in terms of the existence of finite families of open sets that "cover" the space in the sense that each point of the space lies in some set contained in the family. This more subtle notion, introduced by Pavel Alexandrov and Pavel Urysohn in 1929, exhibits compact spaces as generalizations of finite sets. In spaces that are compact in this sense, it is often possible to patch together information that holds locally—that is, in a neighborhood of each point—into corresponding statements that hold throughout the space, and many theorems are of this character.

The term **compact set** is sometimes a synonym for compact space, but usually refers to a compact subspace of a topological space.

13.1.1 Historical development

In the 19th century, several disparate mathematical properties were understood that would later be seen as consequences of compactness. On the one hand, Bernard Bolzano (1817) had been aware that any bounded sequence of points (in the line or plane, for instance) has a subsequence that must eventually get arbitrarily close to some other point, called a limit point. Bolzano's proof relied on the method of bisection: the sequence was placed into an interval that was then divided into two equal parts, and a part containing infinitely many terms of the sequence was selected. The process could then be repeated by dividing the resulting smaller interval into smaller and smaller parts until it closes down on the desired limit point. The full significance of Bolzano's theorem, and its method of proof, would not emerge until almost 50 years later when it was rediscovered by Karl Weierstrass.[1]

In the 1880s, it became clear that results similar to the Bolzano–Weierstrass theorem could be formulated for spaces of functions rather than just numbers or geometrical points. The idea of regarding functions as themselves points of a generalized space dates back to the investigations of Giulio Ascoli and Cesare Arzelà.[2] The culmination of their investigations, the Arzelà–Ascoli theorem, was a generalization of the Bolzano–Weierstrass theorem to families of continuous functions, the precise conclusion of which was that it was possible to extract a uniformly convergent sequence of functions from a suitable family of functions. The uniform limit of this sequence then played precisely the same role as Bolzano's "limit point". Towards the beginning of the twentieth century, results similar to that of Arzelà and Ascoli began to accumulate in the area of integral equations, as investigated by David Hilbert and Erhard Schmidt. For a certain class of Green's functions coming from solutions of integral equations, Schmidt had shown that a property analogous to the Arzelà–Ascoli theorem held in the sense of mean convergence—or convergence in what would later be dubbed a Hilbert space. This ultimately led to the notion of a compact operator as an offshoot of the general notion of a compact space. It was Maurice Fréchet who, in 1906, had distilled the essence of the Bolzano–Weierstrass property and coined the term *compactness* to refer to this general phenomenon (he used the term already in his 1904 paper[3] which led to the famous 1906 thesis).

However, a different notion of compactness altogether had also slowly emerged at the end of the 19th century from the study of the continuum, which was seen as fundamental for the rigorous formulation of analysis. In 1870, Eduard Heine showed that a continuous function defined on a closed and bounded interval was in fact uniformly continuous. In the course of the proof, he made use of a lemma that from any countable cover of the interval by smaller open intervals, it was possible to select a finite number of these that also covered it. The significance of this lemma was recognized by Émile Borel (1895), and it was generalized to arbitrary collections of intervals by Pierre Cousin (1895) and Henri Lebesgue (1904). The Heine–Borel theorem, as the result is now known, is another special property possessed by closed and bounded sets of real numbers.

This property was significant because it allowed for the passage from local information about a set (such as the continuity of a function) to global information about the set (such as the uniform continuity of a function). This sentiment was expressed by Lebesgue (1904), who also exploited it in the development of the integral now bearing his name. Ultimately the Russian school of point-set topology, under the direction of Pavel Alexandrov and Pavel Urysohn, formulated Heine–Borel compactness in a way that could be applied to the modern notion of a topological space. Alexandrov & Urysohn (1929) showed that the earlier version of compactness due to Fréchet, now called (relative) sequential compactness, under appropriate conditions followed from the version of compactness that was formulated in terms of the existence of finite subcovers. It was this notion of compactness that became the dominant one, because it was not only a stronger property, but it could be formulated in a more general setting with a minimum of additional technical machinery, as it relied only on the structure of the open sets in a space.

13.1.2 Basic Examples

Any finite space is trivially compact. A non-trivial example of a compact space is the (closed) unit interval [0,1] of real numbers. If one chooses an infinite number of distinct points in the unit interval, then there must be some accumulation point in that interval. For instance, the odd-numbered terms of the sequence 1, 1/2, 1/3, 3/4, 1/5, 5/6, 1/7, 7/8, ... get arbitrarily close to 0, while the even-numbered ones get arbitrarily close to 1. The given example sequence shows the importance of including the boundary points of the interval, since the limit points must be in the space itself — an open (or half-open) interval of the real numbers is not compact. It is also crucial that the interval be bounded, since in the interval $[0,\infty)$ one could choose the sequence of points 0, 1, 2, 3, ..., of which no sub-sequence ultimately gets arbitrarily close to any given real number.

In two dimensions, closed disks are compact since for any infinite number of points sampled from a disk, some subset of those points must get arbitrarily close either to a point within the disc, or to a point on the boundary. However, an open disk is not compact, because a sequence of points can tend to the boundary without getting arbitrarily close to any point in the interior. Likewise, spheres are compact, but a sphere missing a point is not since a sequence of points can tend to the missing point, thereby not getting arbitrarily close to any point *within* the space. Lines and planes are not compact, since one can take a set of equally-spaced points in any given direction without approaching any point.

13.1.3 Definitions

Various definitions of compactness may apply, depending on the level of generality. A subset of Euclidean space in particular is called compact if it is closed and bounded. This implies, by the Bolzano–Weierstrass theorem, that any infinite sequence from the set has a subsequence that converges to a point in the set. Various equivalent notions of compactness, such as sequential compactness and limit point compactness, can be developed in general metric spaces.

In general topological spaces, however, the different notions of compactness are not equivalent, and the most useful notion of compactness—originally called *bicompactness*—is defined using covers consisting of open sets (see *Open cover definition* below). That this form of compactness holds for closed and bounded subsets of Euclidean space is known as the Heine–Borel theorem. Compactness, when defined in this manner, often allows one to take information that is known locally—in a neighbourhood of each point of the space—and to extend it to information that holds globally throughout the space. An example of this phenomenon is Dirichlet's theorem, to which it was originally applied by Heine, that a continuous function on a compact interval is uniformly continuous; here, continuity is a local property of the function, and uniform continuity the corresponding global property.

13.1.4 Open cover

Definition

Formally, a topological space X is called *compact* if each of its open covers has a finite subcover. That is, X is compact if for every collection C of open subsets of X such that there is a **finite** subset F of C such that Some branches of mathematics such as algebraic geometry, typically influenced by the French school of Bourbaki, use the term *quasi-compact* for the general notion, and reserve the term *compact* for topological spaces that are both Hausdorff and *quasi-compact*. A compact set is sometimes referred to as a *compactum,* plural *compacta*.

13.1.5 Properties of Compact Spaces

Functions and Compact Spaces

A continuous image of a compact space is compact. This implies the extreme value theorem: a continuous real-valued function on a nonempty compact space is bounded above and attains its supremum (Slightly more generally, this is true for an upper semicontinuous function.) As a sort of converse to the above statements, the pre-image of a compact space under a proper map is compact.

Compact Spaces and Set Operations

A closed subset of a compact space is compact, and a finite union of compact sets is compact.

The product of any collection of compact spaces is compact. (This is Tychonoff's theorem, which is equivalent to the axiom of choice.)

Every topological space X is an open dense subspace of a compact space having at most one point more than X, by the Alexandroff one-point compactification. By the same construction, every locally compact Hausdorff space X is an open dense subspace of a compact Hausdorff space having at most one point more than X.

Ordered compact Spaces

A nonempty compact subset of the real numbers has a greatest element and a least element.

Let X be a simply ordered set endowed with the order topology. Then X is compact if and only if X is a complete lattice (i.e. all subsets have suprema and infima).

13.1.6 Examples

- Any finite topological space, including the empty set, is compact. More generally, any space with a finite topology (only finitely many open sets) is compact; this includes in particular the trivial topology.
- Any space carrying the cofinite topology is compact.
- Any locally compact Hausdorff space can be turned into a compact space by adding a single point to it, by means of Alexandroff one-point compactification. The one-point compactification of $\mathbf{R}$ is homeomorphic to the circle $\mathbf{S}^1$; the one-point compactification of $\mathbf{R}^2$ is homeomorphic to the sphere $\mathbf{S}^2$. Using the one-point compactification, one can also easily construct compact spaces which are not Hausdorff, by starting with a non-Hausdorff space.
- The right order topology or left order topology on any bounded totally ordered set is compact. In particular, Sierpiński space is compact.
- No discrete space with an infinite number of points is compact. The collection of all singletons of the space is an open cover which admits no finite subcover. Finite discrete spaces are compact.
- In $\mathbf{R}$ carrying the lower limit topology, no uncountable set is compact.

- In the cocountable topology on an uncountable set, no infinite set is compact. Like the previous example, the space as a whole is not locally compact but is still Lindelöf.
- The closed unit interval [0,1] is compact. This follows from the Heine–Borel theorem. The open interval (0,1) is not compact: the open cover for $n = 3, 4, \ldots$ does not have a finite subcover. Similarly, the set of *rational numbers* in the closed interval [0,1] is not compact: the sets of rational numbers in the intervals cover all the rationals in [0, 1] for $n = 4, 5, \ldots$ but this cover does not have a finite subcover. Here, the sets are open in the subspace topology even though they are not open as subsets of **R**.
- The set **R** of all real numbers is not compact as there is a cover of open intervals that does not have a finite subcover. For example, intervals $(n-1, n+1)$, where n takes all integer values in **Z**, cover **R** but there is no finite subcover.
- On the other hand, the extended real number line carrying the analogous topology *is* compact; note that the cover described above would never reach the points at infinity. In fact, the set has the homeomorphism to [-1,1] of mapping each infinity to its corresponding unit and every real number to its sign multiplied by the unique number in the positive part of interval that results in its absolute value when divided by one minus itself, and since homeomorphisms preserve covers, the Heine-Borel property can be inferred.
- For every natural number *n*, the *n*-sphere is compact. Again from the Heine–Borel theorem, the closed unit ball of any finite-dimensional normed vector space is compact. This is not true for infinite dimensions; in fact, a normed vector space is finite-dimensional if and only if its closed unit ball is compact.
- On the other hand, the closed unit ball of the dual of a normed space is compact for the weak-* topology. (Alaoglu's theorem)
- The Cantor set is compact. In fact, every compact metric space is a continuous image of the Cantor set.
- Consider the set K of all functions $f : \mathbf{R} \to [0,1]$ from the real number line to the closed unit interval, and define a topology on K so

that a sequence in K converges towards if and only if converges towards $f(x)$ for all real numbers x. There is only one such topology; it is called the topology of pointwise convergence or the product topology. Then K is a compact topological space; this follows from the Tychonoff theorem.

- Consider the set K of all functions $f : [0,1] \rightarrow [0,1]$ satisfying the Lipschitz condition $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0,1]$. Consider on K the metric induced by the uniform distance Then by Arzelà–Ascoli theorem the space K is compact.
- The spectrum of any bounded linear operator on a Banach space is a nonempty compact subset of the complex numbers **C**. Conversely, any compact subset of **C** arises in this manner, as the spectrum of some bounded linear operator. For instance, a diagonal operator on the Hilbert space may have any compact nonempty subset of **C** as spectrum.

13.1.7 Algebraic Examples

- Compact groups such as an orthogonal group are compact, while groups such as a general linear group are not.
- Since the p-adic integers are homeomorphic to the Cantor set, they form a compact set.
- The spectrum of any commutative ring with the Zariski topology (that is, the set of all prime ideals) is compact, but never Hausdorff (except in trivial cases). In algebraic geometry, such topological spaces are examples of quasi-compact schemes, "quasi" referring to the non-Hausdorff nature of the topology.
- The spectrum of a Boolean algebra is compact, a fact which is part of the Stone representation theorem. Stone spaces, compact totally disconnected Hausdorff spaces, form the abstract framework in which these spectra are studied. Such spaces are also useful in the study of profinite groups.
- The structure space of a commutative unital Banach algebra is a compact Hausdorff space.
- The Hilbert cube is compact, again a consequence of Tychonoff's theorem.

- A profinite group (e.g., Galois group) is compact.

Check In Progress-I

Q 1. Give Definition of open Cover.

Solution

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Q 2. Define Local Compactness.

Solution

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……..……………………………………………………………………
…………………….…………………………………………………………

13.2 LOCALLY COMPACT SPACE

Compact spaces (especially compact Hausdorff spaces) are extremely "nice" - as we have already studied (optimization problems have solutions; continuous functions are uniformly continuous; integrals exist). There is a more general class of spaces that are important (for example, they include R n) and that arise a lot in analysis (see, for example, the "Riesz representation theorem"). These spaces are too big to be compact, but they are compact when looked at from close-up. More precisely,...

Definition. A space X is locally compact if for each x ∈ X, there exists an open neighborhood U of x with closure U^{-} compact.

When X is also Hausdorff, the property of local compactness becomes much stronger. Let's state this as a theorem.

Notes

Theorem 1. If X is locally compact and Hausdorff, $x \in X$, and U is any neighborhood of x, then there exists a neighborhood V of x such that the closure V^{-} is compact and $V^{-} \subseteq U$.

Remark. So not only does x have some neighborhood with compact closure, it has many; in fact, it has arbitrarily small neighborhoods with compact closure.

The text proves this theorem by first embedding X in its "one-point compactificaton". Instead, let's prove the theorem more directly, and then use this tool to help understand the one-point compactification space. Ultimately, we are all doing the same "dirty work", just changing the order in which we encounter various issues. (And I think the approach in these notes makes the issues clearer.)

Lemma 1.1. If X is Hausdorff, $x \in X$, and C is a compact subset of X with $x \notin C$, then there exist disjoint neighborhoods U(x) and V (C).

Proof. This is stated as in the text. The technique for this proof is something you should know well, useful for other theorems, so here is the proof.

Since X is Hausdorff, for each point $y \in C$, there are disjoint neighborhoods of x and y; let's call these $U_y(x)$ and $V_y(y)$. The set C is covered by $\{V_y : y \in C\}$ and, since C is compact, there is a finite subcover $\{V_{y1}, \ldots, V_{yn}\}$. So $U = U_{y1} \cap \ldots \cap U_{yn}$ and $V = V_{y1} \cup \ldots \cup V_{yn}$ are disjoint neighborhoods of x and C respectively

Lemma 1.2. In a Hausdorff space X, suppose U is a neighborhood of a point x and bd U is compact. Then there exists a neighborhood V of x such that the closure $V^{-} \subseteq U$

Proof. By assumption, bd U is compact. Then, by Lemma (1.1), there exist disjoint neighborhoods W of x and W′ of bd U. Note this implies that the closure W^{-} is disjoint from bdU. Let $V = U \cap W$.

Then $V^{-} \subseteq U^{-} \cap W^{-} = (U \cup \text{bd } U) \cap W^{-} = (U \cap W^{-}) \cup (\text{bd } U \cap W^{-}) = (U \cap W^{-}) \cup \emptyset \subseteq U$.

Remark. The idea in the preceding lemma is that if we can separate x from the boundary of a neighborhood U(x) then we can shrink U to a neighborhood that is "deep" within U, that is the closure of the new neighborhood is contained in U.

Proof of Theorem 1. We have x ∈ U, where U is a given neighborhood of x. By definition of local compactness, there exists a[nother] neighborhood W of x such that the closure W¯ is compact. This makes any closed set contained in W¯ also compact

Consider the set V1 = U ∩ W. We might hope that V1 is the desired neighborhood of x; it certainly is contained in U. But its closure is, in general, not contained in U,. So we have to "trim it down" a little.

The set bd V1 is closed and contained in V¯ 1 ⊆ W¯ which is compact, so bd V1 is compact. By Lemma 1.2, there exists a neighborhood V (x) such that the closure V¯ ⊆ V1; but since V1 = U ∩ W, this says V¯ ⊆ U.

Remark (for the future). Along with finding neighborhoods of a point that lie deep within a given one, we also can use the same kind of thinking (separate points from compact sets, or separate compact sets from each other) to get large families of nested neighborhoods. In fact, we can construct inductively a countable family of neighborhoods of x inside a given U where the countable family is indexed by rationals of the form j 2n for all positive integers j and n, such that the containment relations between the neighborhoods is the same as for the intervals [0, j 2n]. This ultimately lets us construct continuous functions from X to R that "separate points" or "separate points from closed sets". In a [locally] compact Hausdorff space, given two points A, B or a point A and a closed set B missing A, there exists a continuous function f : X → R such that f(x) = 0 and f(a) = 1 for all a ∈ A. This property is sometimes called completely regular. We'll see more theorems like this in later sections.

13.3 COMPACTIFICATION

In mathematics, in general topology, **compactification** is the process or result of making a topological space into a compact space. A compact space is a space in which every open cover of the space contains a finite subcover. The methods of compactification are various, but each is a way of controlling points from "going off to infinity" by in some way adding "points at infinity" or preventing such an "escape".

An Example

Consider the real line with its ordinary topology. This space is not compact; in a sense, points can go off to infinity to the left or to the right. It is possible to turn the real line into a compact space by adding a single "point at infinity" which we will denote by ∞. The resulting compactification can be thought of as a circle (which is compact as a closed and bounded subset of the Euclidean plane). Every sequence that ran off to infinity in the real line will then converge to ∞ in this compactification.

Intuitively, the process can be pictured as follows: first shrink the real line to the open interval $(-\pi,\pi)$ on the x-axis; then bend the ends of this interval upwards (in positive y-direction) and move them towards each other, until you get a circle with one point (the topmost one) missing. This point is our new point ∞ "at infinity"; adding it in completes the compact circle.

A bit more formally: we represent a point on the unit circle by its angle, in radians, going from $-\pi$ to π for simplicity. Identify each such point θ on the circle with the corresponding point on the real line $\tan(\theta/2)$. This function is undefined at the point π, since $\tan(\pi/2)$ is undefined; we will identify this point with our point ∞.

Since tangents and inverse tangents are both continuous, our identification function is a homeomorphism between the real line and the unit circle without ∞. What we have constructed is called the *Alexandroff one-point compactification* of the real line, discussed in more generality below. It is also possible to compactify the real line by adding *two* points, $+\infty$ and $-\infty$; this results in the extended real line.

Definition

An embedding of a topological space *X* as a dense subset of a compact space is called a **compactification** of *X*. It is often useful to embed topological spaces in compact spaces, because of the special properties compact spaces have.

Embeddings into compact Hausdorff spaces may be of particular interest. Since every compact Hausdorff space is a Tychonoff space, and every subspace of a Tychonoff space is Tychonoff, we conclude that any space possessing a Hausdorff compactification must be a Tychonoff space. In fact, the converse is also true; being a Tychonoff space is both necessary and sufficient for possessing a Hausdorff compactification.

The fact that large and interesting classes of non-compact spaces do in fact have compactifications of particular sorts makes compactification a common technique in topology.

Check In Progress

Q 1. If X is Hausdorff, $x \in X$, and C is a compact subset of X with $x \notin C$, then there exist disjoint neighborhoods U(x) and V (C).

Solution

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Q 2. In a Hausdorff space X, suppose U is a neighborhood of a point x and bd U is compact. Then there exists a neighborhood V of x such that the closure $\overline{V} \subseteq U$

Solution

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13.4 STONE–ČECH COMPACTIFICATION

Of particular interest are Hausdorff compactifications, i.e., compactifications in which the compact space is Hausdorff. A topological space has a Hausdorff compactification if and only if it is Tychonoff. In this case, there is a unique (up to homeomorphism) "most general" Hausdorff compactification, the Stone–Čech compactification of *X*, denoted by β*X*; formally, this exhibits the category of Compact Hausdorff spaces and continuous maps as a reflective subcategory of the category of Tychonoff spaces and continuous maps.

"Most general" or formally "reflective" means that the space β*X* is characterized by the universal property that any continuous function from *X* to a compact Hausdorff space *K* can be extended to a continuous function from β*X* to *K* in a unique way. More explicitly, β*X* is a compact Hausdorff space containing *X* such that the induced topology on *X* by β*X* is the same as the given topology on *X*, and for any continuous map $f:X \to K$, where *K* is a compact Hausdorff space, there is a unique continuous map $g:\beta X \to K$ for which *g* restricted to *X* is identically *f*.

The Stone–Čech compactification can be constructed explicitly as follows: let *C* be the set of continuous functions from *X* to the closed interval [0,1]. Then each point in *X* can be identified with an evaluation function on *C*. Thus *X* can be identified with a subset of $[0,1]^C$, the space of *all* functions from *C* to [0,1]. Since the latter is compact by Tychonoff's theorem, the closure of *X* as a subset of that space will also be compact. This is the Stone–Čech compactification.

13.4.1 Spacetime Compactification

Walter Benz and Isaak Yaglom have shown how stereographic projection onto a single-sheet hyperboloid can be used to provide a compactification for split complex numbers. In fact, the hyperboloid is part of a quadric in real projective four-space. The method is similar to that used to provide a base manifold for group action of the conformal group of spacetime.

13.4.2 Projective Space

Real projective space $\mathbf{RP}^n$ is a compactification of Euclidean space $\mathbf{R}^n$. For each possible "direction" in which points in $\mathbf{R}^n$ can "escape", one new point at infinity is added (but each direction is identified with its opposite). The Alexandroff one-point compactification of **R** we constructed in the example above is in fact homeomorphic to $\mathbf{RP}^1$. Note however that the projective plane $\mathbf{RP}^2$ is *not* the one-point compactification of the plane $\mathbf{R}^2$ since more than one point is added.

Complex projective space $\mathbf{CP}^n$ is also a compactification of $\mathbf{C}^n$; the Alexandroff one-point compactification of the plane **C** is (homeomorphic to) the complex projective line $\mathbf{CP}^1$, which in turn can be identified with a sphere, the Riemann sphere.

Passing to projective space is a common tool in algebraic geometry because the added points at infinity lead to simpler formulations of many theorems. For example, any two different lines in $\mathbf{RP}^2$ intersect in precisely one point, a statement that is not true in $\mathbf{R}^2$. More generally, Bézout's theorem, which is fundamental in intersection theory, holds in projective space but not affine space. This distinct behavior of intersections in affine space and projective space is reflected in algebraic topology in the cohomology rings – the cohomology of affine space is trivial, while the cohomology of projective space is non-trivial and reflects the key features of intersection theory (dimension and degree of a subvariety, with intersection being Poincaré dual to the cup product).

Compactification of moduli spaces generally require allowing certain degeneracies – for example, allowing certain singularities or reducible varieties. This is notably used in the Deligne–Mumford compactification of the moduli space of algebraic curves.

Theorem 4. If X is locally compact, Hausdorff and normal and if there is a countable subset $A = \{x1, x2,... \}$ of X such that $ClX\hat{}\ (A) = A \cup \{\infty\}$, then $\beta(X) = X\hat{}$. Proof: By assumption, the point at infinity is the only limit point of A in $X\hat{}$ and therefore A is closed in X. Write $A = B \cup C$, where $B = \{x1, x3, x5,... \}$ and $C = \{x2, x4, x6,... \}$. Since X is normal and B and C are disjoint closed subsets of X, the Urysohn lemma implies there is a continuous $f : X \to [0, 1]$ such that $f(B) = \{0\}$ and $f(C) = \{1\}$. This f is clearly not continuously extendable to $X\hat{}$ and thus $\beta(X) = X\hat{}$.

An immediate consequence of the previous theorem is that if $X\hat{}$ is metrizable, or if X can be written as the countable union of compact sets (e.g., $IR = \cup_{n=1}^{\infty}[-n, n]$), then $\beta(X) = X\hat{}$. It should be pointed out that the converse of the theorem is false. For example, if X is the disjoint union of two copies of SΩ, then the "point at infinity" in $X\hat{}$ will be a point (denoted Ω) joining both copies of SΩ at their "ends". If f is defined from X into the reals by $f(x) = 0$ for all x in one of the copies of SΩ, and $f(x) = 1$ for all x in the other, then certainly f cannot be continuously extended to $X\hat{}$, and hence $\beta(X) = X\hat{}$. But there is no (countable) sequence of points in $X\hat{}$ which can converge to Ω.

The requirement that X be normal is necessary in order to apply the Urysohn lemma. The following lemma allows us to "weaken" that hypothesis to requiring that X be Lindel¨of. [Note that a space X is Lindel¨of if every open covering of X has a countable subcovering.]

Lemma 1. If X is locally compact, Hausdorff and Lindel¨of, then X is normal.

Proof: It is a standard sequence of exercises (see [1], p. 205, exercises 6,7) to show that every locally compact, Hausdorff space is regular and that every regular, Lindel¨of space is normal.

Theorem 5. If X is locally compact, Hausdorff and Lindel¨of, then $\beta(X) = X\hat{}$.

Proof: By theorem 4, we need only show that there is a countable subset A of X such that $ClX\hat{}\ (A) = A \cup \{\infty\}$. Since X is locally compact, for each $x \in X$, there is an open neighborhood Ux of x such that $\bar{U}x = ClX(Ux)$ is compact. Then $U = \{Ux : x \in X\}$ is an open cover of X. Since X is Lindel¨of, there is a countable subcover, say $\{Ux1 , Ux2,... \}$.

For each positive integer n, let $V_n = \cup_{i=1}^{n} U_{x_i}$ (hence, of course, $\bar{V}_n = Cl_X(V_n)$, which is compact) and choose $y_n \in X - \bar{V}_n$. Let $A = \{y_1, y_2, \ldots \}$. If $x \in X - A$, then $x \in U_{x_n}$ for some n, so U_{x_n} is a neighborhood of x which does not contain any of $y_n, y_{n+1}, y_{n+2}, \ldots$. Since X is Hausdorff, we can find a neighborhood of x disjoint from A. Thus $x \in Cl_X(A)$. Now, if C is a compact subset of X, (so that $X - C$ is a neighborhood of ∞ in $\hat{X}$), then there is a V_n such that $C \subseteq V_n$. Thus $\{y_n, y_{n+1}, y_{n+2}, \ldots \} \subseteq (\hat{X} - C)$; i.e., every neighborhood of ∞ in $\hat{X}$, contains points of A. Hence $\infty \in Cl_{\hat{X}}(A)$, or $Cl_{\hat{X}}(A) = A \cup \{\infty\}$.

In general, trying to visualize or understand the Stone-Cech compactification of ˇ a topological space can be mind-boggling if not impossible. We have seen that this is not the case for SΩ. Next we calculate β(SΩ × X) where X is a compact space. First consider C(X, IR) = the collection of all continuous f : X → IR.

13.5 PARACOMPACTNESS

Let X be a topological space.

Definition. The space X is locally compact if each $x \in X$ admits a compact neighborhood N. If X is locally compact and Hausdorff, then all compact sets in X are closed and hence if N is a compact neighborhood of x then N contains the closure the open int(N) around x. Hence, in such cases every point $x \in X$ lies in an open whose closure is compact. Much more can be said about the local structure of locally compact Hausdorff spaces, though it requires some serious theorems in topology (such as Urysohn's lemma) which, while covered in basic topology books, are too much of a digression for us and are not necessary for our purposes. We record one interesting aspect of locally compact spaces:

Lemma. If X is a locally compact Hausdorff space that is second countable, then it admits a countable base of opens $\{U_n\}$ with compact closure.

Proof. Let {Vn} be a countable base of opens. For each x ∈ X there exists an open Ux around x with compact closure, yet some Vn(x) contains x and is contained in Ux. The closure of Vn(x) is a closed subset of the compact Ux, and so V n(x) is also compact. Thus, the Vn's with compact closure are a countable base of opens with compact closure.

Definition. An open covering {Ui} of X refines an open covering {Vj} of X if each Ui is contained in some (perhaps many) Vj .

A simple example of a refinement is a subcover, but refinements allow much greate flexibility: none of the Ui 's needs to be a Vj . For example, the covering of a metric space by open balls of radius 1 is refined by the covering by open balls of radius 1/2. We are interested in special refinements:

Definition. An open covering {Ui} of X is locally finite if every x ∈ X admits a neighborhood N such that N ∩ Ui is empty for all but finitely many i.

For example, the covering of R by open intervals (n−1, n+ 1) for n ∈ Z is locally finite, whereas the covering of (−1, 1) by intervals (−1/n, 1/n) (for n ≥ 1) barely fails to be locally finite: there is a problem at the origin (but nowhere else).

Definition. A topological space X is paracompact if every open coverings admits a locally finite refinement. (It is traditional to also require paracompact spaces to be Hausdorff, as paracompactness is never used away from the Hausdorff setting, in contrast with compactness – though many mathematicians implicity require compact spaces to be Hausdorff too and they reserve a separate word (quasi-compact) for compactness without the assumption of the Hausdorff condition.)

Obviously any compact space is paracompact (as every open cover admits a finite subcover, let alone a locally finite refinement). Also, an arbitrary disjoint union ` Xi of paracompact spaces (given the topology wherein an open set is one that meets each Xi is an open subset) is again paracompact. Note that it is not the case that open covers of a paracompact space admit locally finite subcovers, but rather just locally finite refinements. Indeed, we saw at the outset that Rn is paracompact, but even in the real line there exist open covers with no locally finite subcover: let $U_n = (-\infty, n)$ for $n \geq 1$. All U_n's contain $(-\infty, 0)$, and any subcollection of U_{n_i}'s that covers R has to be infinite since each U_n is "bounded on the right". Thus, no subcover can be locally finite near a negative number.

In general, paracompactness is a slightly tricky property: there are counterexamples that show that an open subset of a paracompact Hausdorff space need not be paracompact. Thus, to prove that an open subset of R^n is paracompact we will have to use special features of R^n. However, just as closed subsets of compact sets are compact, closed subsets of paracompact spaces are paracompact; the argument is virtually the same as in the compact case (extend covers by using the complement of the closed set), so we leave the details to the reader. It is a non-trivial theorem in topology that any metric space is paracompact! This can be found in any introductory topology book, but we will not need it. Our interest in paracompact spaces is due to:

Theorem . Any second countable Hausdorff space X that is locally compact is paracompact.

Proof. Let $\{V_n\}$ be a countable base of opens in X. Let $\{U_i\}$ be an open cover of X for which we seek a locally finite refinement. Each $x \in X$ lies in some U_i and so there exists a $V_{n(x)}$ containing x with $V_{n(x)} \subseteq U_i$. The $V_{n(x)}$'s therefore consistute a refinement of $\{U_i\}$ that is countable. Since the property of one open covering refining another is transitive, we therefore lose no generality by seeking locally finite refinements of countable covers. We can do better: by Lemma 2.2, we can assume that all V_n are compact. Hence, we can restrict our attention to countable covers by opens U_n for which U_n is compact. Since closure commutes with finite unions, by replacing U_n with $\cup_{j \leq n} U_j$ we preserve the compactness condition (as a finite union of compact subsets is compact) and so we can assume that $\{U_n\}$ is an increasing collection of opens with compact closure (with $n \geq 0$). Since U_n is compact yet is covered by the open U_i's, for sufficiently large N we have $U_n \subseteq U_N$. If we recursively replace U_{n+1} with such a U_N for each n, then we can arrange that $U_n \subseteq U_{n+1}$ for each n. Let $K_0 = U_0$ and for $n \geq 1$ let $K_n = U_n - U_{n-1} = U_n \cap$

(X − Un−1), so Kn is compact for every n (as it is closed in the compact Un) but for any fixed N we see that UN is disjoint from Kn for all n > N.

Now we have a situation similar to the concentric shells in our earlier proof of paracompactness of Rn , and so we can carry over the argument from Euclidean spaces as follows. We seek a locally finite refinement of {Un}. For n ≥ 2 the open set Wn = Un+1 − Un−2 contains Kn, so for each x ∈ Kn there exists some Vm ⊆ Wn around x. There are finitely many such Vm's that actually cover the compact Kn, and the collection of Vm's that arise in this way as we vary n ≥ 2 is clearly a locally finite collection of opens in X whose union contains X − U0. Throwing in finitely many Vm's contained in U1 that cover the compact U0 thereby gives an open cover of X that refines {Ui} and is locally finite.

13.6 SUMMARY

We study covering and properties. We study orthocompact and its some examples. We study projective plane and its properties. We study space time compactification.

13.7 KEYWORD

COVERING : Carried out to protect an exposed person from an enemy

ORTHOCOMPACT : is ***orthocompact*** if every open cover has a Q-refinement.

PROJECTIVE : Relating to the unconscious transfer of one's desires or emotions to another person

13.8 QUESTIONS FOR REVIEW

1 Any second countable Hausdorff space X that is locally compact is paracompact.
2 Compact spaces are L−L− compact. Suppose XX is compact; XX is a neighborhood of each of its points implies XX is L−L− compact.

3 The usual real line RR is L−L− compact, since for each x∈Rx∈R, we have x∈(a,b)⊆[a,b]x∈(a,b)⊆[a,b].

Thus [a,b][a,b] is a neighbourhood of xx which is compact by the Heine-Boral theorem. This proves that RR is L−L− compact. But recall that RR is not compact.

4 QQ and QcQc as a subspace of RR are not locally compact.

5 A compact space is L−L− compact.

6 If XX is a Hausdorff locally compact space, then for all x∈Xx∈X and for all neighbourhoods UU of xx, there exists a compact neighbourhood VV of xx such that V⊆UV⊆U.

7 Let f:X→Yf:X→Y be an open continuous surjection. If XX is L−L− compact, the YY is L−L− compact.

8 Local compactness is a closed hereditary property.

9 X1, X2X1, X2 are L−L− compact if and only if X1×X2X1×X2 is L−L− compact.

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13.10 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1.4

Q 2 Check in Section 1

Check in Progress-II

Answer Q. 1 Check in Section 2

Q 2 Check in Section 2

UNIT 14: COVERING SPACE AND UNIFORM SPACE

STRUCTURE

14.0 OBJECTIVE

After study this topic we able to learn

* Introduction covering space

* Learn Uniform space

* Learn product and subspace

* Learn Compactifications

* Learn Total Boundedness and Uniform weight

14.1 INTRODUCTION COVERING SPACE

Given a topological space X, we're interested in spaces which "cover" X in a nice way. Roughly speaking, a space Y is called a covering space of X if Y maps onto X in a locally homeomorphic way, so that the pre-image of every point in X has the same cardinality. It turns out that the covering spaces of X have a lot to do with the fundamental group of X. The subgroups of π1(X) correspond exactly to the connected covering spaces of X. Also, for nice enough spaces X, there's a special covering space called the universal cover, on which π1(X) acts. Covering spaces are important not just for algebraic topology but also for differential geometry, Lie groups, Riemann surfaces, geometric group theory . . . In mathematics, more specifically algebraic topology, a **covering map** (also **covering projection**) is a continuous function [1] from a topological space to a topological space such that each point in has an open neighbourhood **evenly covered** by (as shown in the image); the precise definition is given below. In this case, is called a **covering space** and the **base space** of the covering projection. The definition implies that every covering map is a local homeomorphism.

Covering spaces play an important role in homotopy theory, harmonic analysis, Riemannian geometry and differential topology. In Riemannian geometry for example, ramification is a generalization of the notion of covering maps. Covering spaces are also deeply intertwined with the study of homotopy groups and, in particular, the fundamental group. An important application comes from the result that, if is a "sufficiently good" topological space, there is a bijection between the collection of all isomorphism classes of connected coverings of and the conjugacy classes of subgroups of the fundamental group of .

14.1.1 Definition and Basic Examples

Throughout, all spaces are topological spaces and all maps are continuous.

Definition 1. A covering space or cover of a space X is a space Xe together with a map p : Xe → X satisfying the following condition: every point x ∈ X has an open neighborhood Ux ⊆ X such that p −1 (Ux) is a disjoint union of open sets, each of which is mapped by p homeomorphically onto Ux. You can visualize the pre-image of the neighborhood Ux as a “stack of pancakes”, each pancake being homeomorphic to Ux.

Some more terminology: sometimes the space X is called the base space, the map p is called the covering map or projection, and the pre-image p −1 (x) of some point x in the base space is called the fiber over x.

Examples.

1. There’s always the trivial cover: a space covers itself, with the covering map being the identity map.

2. The map p : R → S 1 given by p(t) = e it is a covering map, wrapping the real line round and round the circle. The pre-image of a little open arc in the circle is a collection of open intervals in the real line, offset by multiples of 2π.

3. Another cover of the circle is the map p : S 1 → S 1 given by p(z) = z n , where n is a positive integer. This wraps the circle around itself n times.

4. Consider the equivalence relation on R 2 given by (x, y) ∼ (x + m, y + n), where m and n are any integers. Let p : R 2 → R 2/∼ be the quotient map. Then the image of p is the torus obtained by identifying opposite sides of a square, and p is a covering map.

5. The real projective plane RP 2 can be thought of in several equivalent ways: as the set of lines through the origin in R 3 , as S 2 with the equivalence relation x ∼ −x, and as the set of non-zero points of R 3 with the equivalence relation x ∼ λx, where λ is a non-zero scalar. If we select

the second way of thinking about RP 2 , then S 2 is a covering space for RP 2 , with the covering map being the quotient map.

6. The figure-of-eight graph has lots of covering spaces, and I'll draw some of them on the board.

Given a neighborhood Ux in the base space, the fiber over each point in Ux must have the same cardinality. So, if the base space is connected, this cardinality is constant over the whole space. The cardinality of each fiber is then called the number of sheets of the covering. The cover of S 1 in Example 3 has n sheets, while the cover of RP 2 by S 2 is a two-sheeted covering.

14.1.2 Liftings

In this section p : Xe → X is always a covering space.

A lift of a map f : Y → X is a map ˜f : Y → Xe such that p◦ ˜f = f. There are several key results about existence and uniqueness of liftings, and these have important applications.

For instance, since a covering space is a topological space, it has a fundamental group. The following proposition relates the fundamental group of a covering space to the fundamental group of the base space, and is proved using liftings of homotopies.

Proposition 2. Fix basepoints x0 ∈ X and xe0 ∈ p −1 (x0). Then the homomorphism

p∗ : π1(X, e xe0) → π1(X, x0)

is injective. So, we may identify π1(X, e xe0) with the subgroup p∗(π1(X, e xe0)) of π1(X, x0). The choice of basepoint does matter here: different choices of xe0 in the fiber over x0 will yield conjugate subgroups of π1(X, x0).

Examples

1. Let p : R → S 1 be the covering map p(t) = e it and, for each positive integer n, let pn : S 1 → S1 be the covering map pn(z) = z n . Then

$\pi_1(R, 0)$ is trivial, so its image under p_* is the trivial subgroup of $\pi(S^1, 1) = Z$. The image of $(p_n)_*$ is the subgroup nZ of Z. 2. Let $p : S^2 \to RP^2$ be the covering map which identifies antipodes. Since S^2 has trivial fundamental group, the image under p_* is also trivial.

2. The fundamental group of a covering space which is a graph can be calculated using the Seifert–Van Kampen Theorem. You can then use Proposition 2 to show that, for instance, the free group on two generators has subgroups which are free on three generators, and on countably many generators.

Another important result on liftings concerns liftings of paths in the base space.

Proposition 3. Let $f : I \to X$ be a path with starting point $f(0) = x_0$. Then for each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$ so that $\tilde{f}(0) = \tilde{x}_0$.

In particular, once we fix a basepoint x_0 in X, then for each $\tilde{x}_0 \in p^{-1}(x_0)$, every loop in X based at x_0 has a unique lift to a path in the covering space $\tilde{X}$ starting at $\tilde{x}_0$. This is used to prove the following result.

Proposition 4. When X and $\tilde{X}$ are path-connected, the number of sheets of the covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

14.1.3 Maps between Covering Spaces

Suppose $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are two covering spaces. A homomorphism of covering spaces is a map $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$. An isomorphism of covering spaces is an invertible map (that is, homeomorphism) $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$.

An isomorphism from a covering space to itself is sometimes called a deck transformation or covering transformation (think of shuffling a deck of cards). Deck transformations permute fibers. The set of deck

Notes

transformations of a covering space forms a group under composition. By a unique lifting property, a deck transformation is completely determined by where it sends a single point.

Examples.

1. Each translation of the real line by an integer multiple of 2π is a deck transformation of the covering space p : R → S 1 , where p(t) = e it. The group of deck transformations is isomorphic to Z.

   2. Rotating the circle S 1 by an integer multiple of 2π/n is a deck transformation of the covering space z 7→ z n . The group of deck transformations is cyclic of order n.

   3. Each translation of R2 by a vector (m, n), where m and n are integers, is a deck transformation of the covering space of the torus. The group of deck transformations is isomorphic to Z 2 .

- Every space trivially covers itself.
- A connected and locally path-connected topological space has a universal cover if and only if it is semi-locally simply connected.

- is the universal cover of the unit circle .
- The spin group is a double cover of the special orthogonal group and a universal cover when . The accidental, or exceptional isomorphisms for Lie groups then give isomorphisms between spin groups in low dimension and classical Lie groups.

- The unitary group has universal cover .
- The n-sphere is a double cover of real projective space and is a universal cover for .
- Every manifold has an orientable double cover that is connected if and only if the manifold is non-orientable.
- The uniformization theorem asserts that every Riemann surface has a universal cover conformally equivalent to the Riemann sphere, the complex plane, or the unit disc.
- The universal cover of a wedge of circles is the Cayley graph of the free group on generators, i.e. a Bethe lattice.

- The torus is a double cover of the Klein bottle. This can be seen using the polygon's for the torus and the Klein bottle, and observing that the double cover of the circle (embedding into sending).
- Every graph has a bipartite double cover. Since every graph is homotopic to a wedge of circles, its universal cover is a Cayley graph.
- Every immersion from a compact manifold to a manifold of the same dimension is a covering of its image.
- Infinite-fold abelian covering graphs of finite graphs are regarded as abstractions of crystal structures.[6]
- Another effective tool for constructing covering spaces is using quotients by free finite group actions.
- For example, the space defined by the quotient of (embedded into) is defined by the quotient space via the -action . This space, called a lens space, has fundamental group and has universal cover .

For instance the diamond crystal as an abstract graph is the maximal abelian covering graph of the dipole graph D_4

- The map of affine schemes forms a covering space with as its group of deck transformations. This is an example of a cyclic Galois cover.

14.1.4 G-Coverings

Let *G* be a discrete group acting on the topological space *X*. This means that each element *g* of *G* is associated to a homeomorphism H_g of *X* onto itself, in such a way that $H_{g\,h}$ is always equal to $H_g \circ H_h$ for any two elements *g* and *h* of *G*. (Or in other words, a group action of the group *G* on the space *X* is just a group homomorphism of the group *G* into the group Homeo(*X*) of self-homeomorphisms of *X*.) It is natural to ask under what conditions the projection from *X* to the orbit space *X*/*G* is a covering map. This is not always true since the action may have fixed points. An example for this is the cyclic group of order 2 acting on a product $X \times X$ by the twist action where the non-identity element acts by $(x, y) \mapsto (y, x)$. Thus the study of the relation between the fundamental groups of *X* and *X*/*G* is not so straightforward.

However the group *G* does act on the fundamental groupoid of *X*, and so the study is best handled by considering groups acting on groupoids, and the corresponding *orbit groupoids*. The theory for this is set down in Chapter 11 of the book *Topology and groupoids* referred to below. The main result is that for discontinuous actions of a group *G* on a Hausdorff space *X* which admits a universal cover, then the fundamental groupoid of the orbit space *X*/*G* is isomorphic to the orbit groupoid of the fundamental groupoid of *X*, i.e. the quotient of that groupoid by the action of the group *G*. This leads to explicit computations, for example of the fundamental group of the symmetric square of a space.

14.2 THE UNIVERSAL COVER AND SUBGROUPS OF THE FUNDAMENTAL GROUP

We saw that the induced homomorphism from the fundamental group of a covering space to the fundamental group of the base space is injective. This leads to the question: can every subgroup of π1(X, x0) be realized as p∗(π1(Xe0, xe0)) for some covering space p : Xe → X and xe0 ∈ p −1 (x0)? It turns out that the answer is yes if X is a reasonably nice space (path-connected, locally path-connected and semilocally simply connected, to be precise).

To prove this, you first construct a universal cover : that is, a covering space Xe of X which is simply connected. The universal cover is unique up to isomorphism.

Examples. The universal cover of the circle is the real line, of the torus is R 2 , of RP 2 is the sphere S 2 , and of the figure-of-eight graph is the infinite 4-valent tree.

Since the universal cover Xe is simply connected, π1(X, e xe0) is trivial, so its image under p∗ is the trivial subgroup of π1(X, x0). To realize all the other subgroups of π1(X, x0), you take quotients of the universal cover.

Another important feature of the universal cover is that the fundamental group of the base space acts on the universal cover by deck transformations. The action is determined as follows. Take a basepoint x0 ∈ X and a preimage xe0 of x0 in the universal cover Xe. Then each element of π1(X, x0) is represented by a loop f : I → X based at x0. There is a unique lift ˜f : I → Xe starting at xe0. Then we define the action of the homotopy class [f] on xe0 by

$$[f] \cdot xe0 = \tilde{}f(1).$$

The quotient under this group action is the base space.

The universal cover of a connected topological space X is a simply connected space Y with a map $f : Y \to X$ that is a covering map. If X is simply connected, i.e., has a trivial fundamental group, then it is its own universal cover. For instance, the sphere $\mathbb{S}^2$ is its own universal cover. The universal cover is always unique and, under very mild assumptions, always exists. In fact, the universal cover of a topological space X exists iff the space X is connected, locally pathwise-connected, and semilocally simply connected.

Any property of X can be lifted to its universal cover, as long as it is defined locally. Sometimes, the universal covers with special structures can be classified. For example, a Riemannian metric on X defines a metric on its universal cover. If the metric is flat, then its universal cover is Euclidean space. Another example is the complex structure of a Riemann surface X, which also lifts to its universal cover. By the uniformization theorem, the only possible universal covers for X are the open unit disk, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.

The above left diagram shows the universal cover of the torus, i.e., the plane. A fundamental domain, shaded orange, can be identified with the torus. The real projective plane is the set of lines through the origin, and its universal cover is the sphere, shown in the right figure above. The only nontrivial deck transformation is the antipodal map.

The compact Riemann surfaces with genuses $g > 1$ are g-holed tori, and their universal covers are the unit disk. The figure above shows a hyperbolic regular octagon in the disk. With the colored edges identified, it is a fundamental domain for the double torus. Each hole has two loops, and cutting along each loop yields two edges per loop, or eight edges in total. Each loop is also shown in a different color, and arrows are drawn to provide instructions for lining them up. The fundamental domain is in gray and can be identified with the double torus illustrated below. The above animation shows some translations of the fundamental domain by deck transformations, which form a Fuchsian group. They tile the disk by analogy with the square tiling the plane for the square torus.

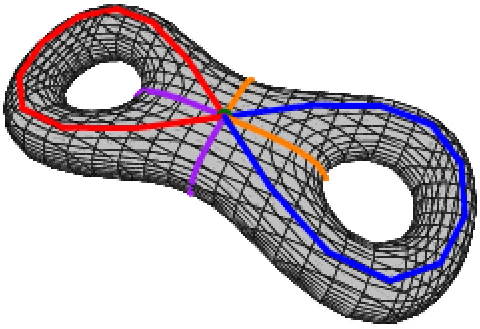

Figure 2.1

Although it is difficult to visualize a hyperbolic regular octagon in the disk as a cut-up double torus, the illustration above attempts to portray this. It is unfortunate that no hyperbolic compact manifold with constant negative curvature, can be embedded in $\mathbb{R}^3$. As a result, this picture is not isometric to the hyperbolic regular octagon. However, the generators for the fundamental group are drawn in the same colors, and are examples of so-called cuts of a Riemann surface.

Roughly speaking, the universal cover of a space is obtained by the following procedure. First, the space is cut open to make a simply connected space with edges, which then becomes a fundamental domain, as the double torus is cut to become a hyperbolic octagon or the square torus is cut open to become a square. Then a copy of the fundamental domain is added across an edge. The rule for adding a copy across an edge is that every point has to look the same as the original space, at least nearby. So the copies of the fundamental domain line up along edges which are identified in the original space, but more edges may also line up. Copies of the fundamental domain are added to the resulting space recursively, as long as there remains any edges. The result is a covering map with possibly infinitely many copies of a fundamental domain which is simply connected.

Any other covering map of X is in turn covered by the universal cover of X, $\tilde{X}$. In this sense, the universal cover is the largest possible cover. In rigorous language, the universal cover has a universal property.

If $p : A \to X$ is a covering map, then there exists a covering map $p' : \tilde{X} \to A$ such that the composition of p and $\tilde{p}$ is the projection from the universal cover to X.

Check In Progress

Q. 1 Define G-Covering.

Solution

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Q. 2 Define Uniform cover & Univarsal space.

Solution

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14.2.2 Uniform Space

A *uniform space* is a set with a uniform structure defined on it. A uniform structure (a uniformity) on a space X is defined by the specification of a system A of subsets of the product X×X. Here the system A must be a filter (that is, for any V1,V2 the intersection V1∩V2 is also contained in A, and if W⊃V, V∈A, then W∈A) and must satisfy the following axioms:

U1) every set V∈A contains the diagonal Δ={(x,x)|x∈X};

U2) if V∈A, then V−1={(y,x)|(x,y)∈V}∈A;

U3) for any V∈A there is a W∈A such

that W∘W⊂Vhere W∘W={(x,y)| there is

a z∈X with (x,z)∈W,(z,y)∈W}W∘W={(x,y)| there is

a z∈X with (x,z))∈W}.

The elements of A are called entourages of the uniformity defined by A.

A uniformity on a set X can also be defined by the specification of a system of coverings C on X satisfying the following axioms:

C1) if α∈C and αα refines a covering β, then β∈C;

C2) for any α1,α2∈C, there is a covering β∈C that star-refines both α1 and α2 (that is, for any x∈X all elements of β containing xx lie in certain elements of α1and α2.

Coverings that belong to C are called uniform coverings of X (relative to the uniformity defined by C).

These two methods of specifying a uniform structure are equivalent. For example, if the uniform structure on XX is given by a system of entourages A, then a system of uniform coverings CC of XX can be constructed as follows. For each V∈Athe family α(V)={V(x)|x∈V} (where V(x)={y|(x,y)∈V} is a covering of X. A covering αα belongs to C if and only if αα can be refined by a covering of the form α(V), V∈A. Conversely, if C is a system of uniform coverings of a uniform space, a system of entourages is formed by the sets of the form U={H×H|H∈α}U={H×H|H∈α}, α∈Cα∈C, and all the sets containing them.

A uniform structure on XX can also be given via a system of pseudo-metrics (cf. Pseudo-metric). Every uniformity on a set XX generates a topology T={G⊂X| for any x∈G there is a V∈A such that V(x)⊂G}T={G⊂X| for any x∈G there is a V∈A such that V(x)⊂G}.

The properties of uniform spaces are generalizations of the uniform properties of metric spaces (cf. Metric space). If (X,ρ) is a metric space, then on X there is a uniformity generated by the metric ρρ. A system of entourages for this uniformity is formed by all sets containing sets of the form {(x,y)|ρ(x,y)<ε}{(x,y)|ρ(x,y)<ε}, ε>0ε>0. Here the topologies on X induced by the metric and the uniformity coincide. Uniform structures generated by metrics are called metrizable.

Uniform spaces were introduced in 1937 by A. Weil [We] (by means of entourages; the definition of uniform spaces by means of uniform coverings was given in 1940, see [Tu]). However, the idea of the use of multiple star-refinement for the construction of functions appeared earlier with L.S. Pontryagin (see [Po]) (afterwards this idea was used in the proof of complete regularity of the topology of a separable uniform space). Initially, uniform spaces were used as tools for the study of the topologies (generated by them) (similar to the way a metric on a metrizable space was often used for the study of the topological properties of the space). However, the theory of uniform spaces is of independent interest, although closely connected with the theory of topological spaces.

A mapping f:X→Y from a uniform space X into a uniform space Y is called uniformly continuous if for any uniform covering αα of Y the system f−1α={f−1U|U∈α} is a uniform covering of X. Every uniformly-continuous mapping is continuous relative to the topologies generated by the uniform structures on X and Y. If the uniform structures on X and Y are induced by metrics, then a uniformly-continuous mapping f:X→Y turns out to be uniformly continuous in the classical sense as a mapping between metric spaces (cf. Uniform continuity).

Of more interest is the theory of uniform spaces that satisfy the additional axiom of separation:

U4) ⋂V∈AV=Δ⋂ (in terms of entourages), or

C3) for any two points x,y∈X,, x≠y, there is an α∈C such that no element of αα simultaneously contains xx and yy (in terms of uniform coverings). From now on only uniform spaces equipped with a separating uniform structure will be considered. The topology on X generated by a separating uniformity is completely regular and, conversely, every completely-regular topology on Xis generated by some separating uniform structure. As a rule, there are many different uniformities generating the same topology on X. In particular, a metrizable topology can be generated by a non-metrizable separating uniformity.

A uniform space (X,A) is metrizable if and only if A has a countable base. Here, a base of a uniformity is (in terms of entourages) any subsystem B⊂A satisfying the condition: For any V∈A there is a $W\in B$ such that $W\subset V$, or (in terms of uniform coverings) a subsystem $A\subset C$ such that for any $\alpha\in C$ there is a $\beta\in A$ that refines α. The weight of a uniform space (X,A) is the least cardinality of a base of the uniformity A.

Let MM be a subset of a uniform space (X,A). The system of entourages $AM=\{(M\times M)\cap V|V\in A\}\}$ defines a uniformity on M. The pair (M,AM) is called a subspace of (X,A). A mapping $f:X\to Y$ from a uniform space (X,A)(X,A) into a uniform space (Y,A′) is called a uniform imbedding if ff is one-to-one and uniformly continuous and if $f-1:(f(X,A'fX)\to(X,A)$ is also uniformly continuous.

A uniform space X is called complete if every Cauchy filter in X (that is a filter containing some element of each uniform covering) has a cluster point (that is, a point lying in the intersection of the closures of the elements of the filter). A metrizable uniform space is complete if and only if the metric generating its uniformity is complete. Any uniform space (X,A) can be uniformly imbedded as an everywhere-dense subset in a unique (up to a uniform isomorphism) complete uniform space (X~,A~), which is called the completion of (X,A)(X,A). The topology of the completion (X~,A~) of a uniform space (X,A) is compact if and only if AA is a pre-compact uniformity (that is, such that any uniform covering refines to a finite uniform covering). In this case the space X~ is a compactification of X and is called the Samuel extension of X relative to the uniformity A. For each compactification bX of X there is a unique pre-compact uniformity on X whose Samuel extension coincides with bX. Thus, all compactifications can be described in the language of pre-compact uniformities. On a compact space there is a unique uniformity (complete and pre-compact).

Every uniformity A on a set X induces a proximity δ by the following formula:

$$A\delta B\Leftrightarrow(A\times B)\cap V\neq\emptyset$$

for all $V\in A$. Here the topologies generated on X by the uniformity A and the proximity δδ coincide. Any uniformly-continuous mapping is proximity continuous relative to the proximities generated by the uniformities. As a rule, there are many different unifo mities generating the same proximity on X By the same token, the set of uniformities on X decomposes into equivalence classes (two uniformities are equivalent if the proximities they induce coincide). Each equivalence class of uniformities contains precisely one pre-compact uniformity; moreover, the Samuel extensions relative to these uniformities coincide with the Smirnov extensions (see Proximity space) relative to the proximity induced by the uniformities of the class. There is a natural partial order on the set of uniformities on XX: A>A″ if A⊃A′. Among all uniformities on X generating a fixed topology there is a largest, the so-called universal uniformity. It induces the Stone–Čech proximity on X. Every pre-compact uniformity is the smallest element in its equivalence class. If C is the system of uniform coverings of some uniformity on XX, then the system of uniform coverings of the equivalent pre-compact uniformity consists of those coverings of XX that refine a finite covering from CC.

The product of uniform spaces (Xt,At)(Xt,At), t∈Tt∈T, is the uniform space (∏Xt,∏At)(∏Xt,∏At), where ∏At∏At is the uniformity on ∏Xt∏Xt with as base for the entourages sets of the form

{({xt},{yt})|(xti,yti)∈Vti,i=1,…,n},{({xt},{yt})|(xti,yti)∈Vti,i=1,…,n},

ti∈T,Vti∈Ati,i=1,2,…ti∈T,Vti∈Ati,i=1,2,…

The topology induced on ∏Xt∏Xt by the uniformity ∏At∏At coincides with the Tikhonov product of the topologies of the spaces XtXt. The projections of the product onto the components are uniformly continuous. Every uniform space of weight ττ can be imbedded in a product of ττ copies of a metrizable uniform space.

The topology of a metrizable uniform space is paracompact, by Stone's theorem. However, Isbell's problem on the uniform paracompactness of metrizable uniform spaces has been solved negatively. An example of a metrizable uniform space having a uniform covering with no locally finite uniform refinement has been constructed [Sh].

In the dimension theory of uniform spaces, the uniform dimension invariants δdδd and ΔdΔd, defined by analogy with the topological dimension dimdim (δdδd using finite uniform coverings and ΔdΔd using all uniform coverings), and the uniform inductive dimension δIndδInd are basic. The dimension δIndδInd is defined by analogy with the large inductive dimension IndInd, by induction relative to the dimensions of proximity partitions between distant (in the sense of the proximity induced by the uniformity) sets. Here, a set HH is called a proximity partition between AA and BB (where AδBAδB) if for any δδ-neighbourhood UU of HH such that U∩(A∪B)≠∅U∩(A∪B)≠∅ one

Various generalizations of uniform spaces have been obtained by weakening the axioms of a uniformity. Thus, in the axiomatics of a quasi-uniformity (see [Cs]) the symmetry axiom is excluded. For the definition of a generalized uniformity (see [Ku]) (an ff-uniformity), uniform families of subsets of X, which in general are not coverings, are used instead of uniform coverings (most of these families turn out to be everywhere-dense in the topology generated by the ff-uniformity). One of the generalizations of a uniformity — the so-called θθ-uniformity — is connected with the presence of the topology on a uniform space.

14.2.2 Comments

Pre-compact uniform spaces are also called totally bounded, and universal uniformities are also called fine uniformities.

Another description of k-spaces is as follows: A Hausdorff space X is a kk-space if and only if it satisfies the following condition: A subset of X is closed in X if and only if its intersection with every compact subset of X is closed.

The construction of a metrizable uniform space that is not uniformly paracompact (i.e. has no base of (uniformly) locally finite uniform coverings) was done independently by E.V. Shchepin [Sh] and J. Pelant [Pe]. In [Pe] it is also shown that in some models of set theory (ZFC), the uniform coverings of a uniform space of power at most 1 need not form a base for a uniformity.

Check In Progress

Q. 1 Define Uniform Space.

Solution

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Q. 2 Define Uniform cover & Univarsal space.

Solution

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14.2.3 Uniform Properties

When trying to generalize metric concepts to wider classes of spaces one encounters the countability barrier: almost no non-trivial uncountable construction preserves metrizability. The category of uniform spaces and uniformly continuous maps provides a convenient place to carry out these generalizations.

Below we invariably let X be our uniform space, with U its family of entourages and U the family of uniform covers.

14.2.3.1 Uniform Continuity

A map $f :(X,U) \to (Y,V)$ between uniform spaces is uniformly continuous if $(f \times f)^{-1} [V] \in U$ whenever $V \in V$, equivalently, if $\{ f^{-1} [A]: A \in A\} \in U$ whenever $A \in V$. A uniformly continuous map is also continuous with respect to the uniform topologies and the converse is, as in the metric case, true for compact Hausdorff spaces.

A bijection that is uniformly continuous both ways is a uniform isomorphism. A uniform property then is a property of uniform spaces that is preserved by uniform isomorphisms.

14.2.3.2 Products and subspaces

It is straightforward to define a uniform structure on a subset Y of a uniform space X: simply intersect the entourages with Y × Y (or trace the uniform covers on Y). To define a product uniformity one may follow the construction of the product topology and define a subbase for it. Given a family $\{(X_i,U_i)\}_{i\in I}$ of uniform spaces define a family of entourages in the square of Q i Xi using the projections π_i : $\{(\pi_i \times \pi_i)^{-1}[U]: U \in U_i, i \in I\}$.

These constructions have the right categorical properties, so that we obtain subobjects and products in the category of uniform spaces and uniformly continuous maps. The uniform topology derived from the product and subspace uniformities are the product and subspace topologies derived from the original uniform topologies, respectively.

14.2.3.3 Uniform Quotients

A map $q : X \to Q$ between uniform spaces is a uniform quotient map if it is onto and has the following universality property: whenever $f : Q \to Y$ is a map to a uniform space Y then f is uniformly continuous if $f \circ q$ is. Every uniformly continuous map $f : X \to Y$ admits a factorization $f = f0 \circ q$ with q a uniform quotient map and f0 a (uniformly continuous) injective map.

In analogy with the topological situation one can, given a surjection f from a uniform space X onto a set Y , define the quotient uniformity on Y to be the finest uniformity that makes f uniformly continuous. The resulting map is uniformly quotient and all uniform quotient maps arise in this way.

The uniform topology of a quotient uniformity is not always the quotient topology of the original uniform topology: if X is completely regular but not normal, as witnessed by the disjoint closed sets F and G, then identifying F to one point results in a space which is Hausdorff but not (completely) regular, hence the quotient uniformity from the fine uniformity (see below) does not generate the quotient topology. There is even a canonical construction that associates to every separated uniform space X a uniform space Y with a uniform quotient map $q : X \to Y$ and such that the uniform topology of Y is discrete.

Uniform quotient maps behave different from topological quotient maps in other respects as well: every product of uniform quotient maps is again a uniform quotient map.

14.2.3.4 Completeness

We say that X is complete (and U or U a complete uniformity) if every Cauchy filter converges. A filter F is Cauchy if for every V ∈ U there is F ∈ F with F × F ⊆ V or, equivalently, if F ∩ A 6= ∅ for all A ∈ U. Closed subspaces and products of complete spaces are again complete.

Every uniform space has a completion, this is a complete uniform space that contains a dense and uniformly isomorphic copy of X. As underlying set of a completion one can take the set eX of minimal Cauchy filters. Every entourage U of U is extended to Ue = {(F,G): (∃F ∈ F)(∃G ∈ G)(F × G ⊆ U)}; the family {Ue: U ∈ U} generates a complete uniformity on eX. If x ∈ X then its neighbourhood filter Fx is a minimal Cauchy filter and x 7→ Fx is a uniform embedding.

As in the case of metric completion the completion of a separated uniform space is unique up to uniform isomorphism.

Using the canonical correspondence between nets and filters (see the article on Convergence) one can define a Cauchy net to be a net whose associated filter is Cauchy. This is equivalent to a definition more akin to that of a Cauchy sequence: A net (tα)α∈D is Cauchy if for every entourage U there is an α such that (tβ,tγ) ∈ U whenever β, γ > α.

14.2.3.5 Total Boundedness

We say X is totally bounded or precompact if for every entourage U (or uniform cover A) there is a finite set F such that U[F] = X (or St(F,A) = X). Subspaces and products of precompact spaces are again precompact.

A metrizable space has a compatible totally bounded metric iff it is separable. A uniformizable space always has a compatible totally bounded uniformity; indeed, for any uniform space (X,U) the family of all finite uniform covers is a base for a (totally bounded) uniformity pU with the same uniform topology, this uniformity is the precompact reflection of U.

The metric theorem that equates compactness with completeness plus total boundedness remains valid in the uniform setting; likewise a Tychonoff space is compact if every compatible uniformity is complete. It is not true that a Tychonoff space is compact iff every compatible uniformity is totally bounded. The ordinal space $\omega 1$ provides a counterexample: it is not compact and it has only one compatible uniformity (the family of all neighbourhoods of the diagonal), which necessarily is totally bounded.

14.2.3.6 Uniform Weight

The weight, w(X,U), is the minimum cardinality of a base. A uniformity U can be generated by κ pseudometrics iff w(X,U) 6 $\kappa \cdot \aleph 0$ iff the separated quotient bX admits a uniform embedding into a product of κ many (pseudo)metric spaces (with its product uniformity). In particular: a uniformity is generated by a single pseudometric iff its weight is countable.

The uniform weight u(X) of a Tychonoff space X is the minimum weight of a compatible uniformity. This is related to other cardinal functions by the inequalities u(x) 6 w(X) 6 u(X)·c(X). The first follows by considering a compactification of the same weight as X, the second from the fact that each pseudometric contributes a σ -discrete family of open sets to a base for the open sets. The uniform weight is related to the metrizability degree: m(X) is the minimum κ such that X has an open base that is the union of κ many discrete families, whereas u(X) is the minimum κ such that X has an open base that is the union of κ many discrete families of cozero sets. Thus m(X) 6 u(X); equality holds for normal spaces and is still an open problem for Tychonoff spaces.

14.2.3.7 Fine Uniformities

Every family {Ui}i of uniformities has a supremum W i Ui . In terms of entourages it is generated by the family of all finite intersections of elements of S i Ui , i.e., S i Ui is used as a subbase. If all the Ui are compatible with a fixed topology T then so is W i Ui . This implies that every Tychonoff space admits a finest uniformity, the fine uniformity or universal uniformity, it is the one generated by the family of all normal covers or by the family of all pseudometrics d f defined above. The fine uniformity is denoted U f .

One says that a uniformity U itself is fine (or a topologically fine uniformity) if it is the fine uniformity of its uniform topology τU .

The equivalence of full normality and paracompactness combined with the constructions of pseudometrics described above yield various characterizations of the covers that belong to the fine uniformity: they are the covers that have locally finite (or σ -locally finite or σ -discrete) refinements consisting of cozero sets. From this it follows that the precompact reflection of the fine uniformity is generated by the finite cozero covers.

Continuity Versus Uniform Continuity

Every continuous map from a fine uniform space to a uniform space (or pseudometric space) is uniformly continuous; this property characterizes fine uniform spaces. Uniform spaces on which every continuous real-valued function is uniformly continuous are called UC-spaces. A metric UCspace is also called an Atsuji space.

The precompact reflection of a fine uniformity yields a space where all bounded continuous real-valued functions are uniformly continuous, these are also called BU-spaces.

14.3 COMPACTIFICATIONS

There is a one-to-one correspondence between the families of compactifications of a Tychonoff space and the compatible totally bounded uniformities. If γ X is a compactification of X then the uniformity that X inherits from γ X is compatible and totally bounded. Conversely, if U is a compatible totally bounded uniformity on X then its completion is a compactification of X, the Samuel compactification of (X,U). The correspondence is order-preserving: the finer the uniformity the larger the compactification. Consequently the compactification that corresponds to the precompact reflection Ufin of the fine uniformity is exactly the Cech–Stone ˇ compactification. It also follows that a space has exactly one compatible uniformity iff it is almost compact.

14.3.1 Proximities

There is also a one-to-one correspondence between the proximities and precompact uniformities.

Indeed, a uniformity U determines a proximity δU by A δU B iff U[A] ∩ U[B] 6= ∅ for every entourage U (intuitively: proximal sets have distance zero).

Conversely, a proximity δ determines a uniformity Uδ : the family of sets X 2 \ (A × B) with A 6δ B forms a subbase for Uδ . This uniformity is always precompact and, in fact, UδU is the precompact reflection of U.

The Samuel compactification of (X,Uδ) is also known as the Smirnov compactification of (X, δ)

14.3.2 Function Spaces

Uniformities also allow one to formulate and prove theorems on uniform convergence and continuity in a general setting. Thus, given a uniform space (Y,V) and a set (or space) X one can define various uniformities on the set Y X of all maps from X to Y . Let A be a family of subsets of X. For V ∈ V and A ∈ A one defines the entourage EA,V to be the set {(f, g): (∀x ∈ A)((f (x), g(x)) ∈ V)}. The family of sets EA,V serves as a subbase for a uniformity. The corresponding uniform topology is called the topology of uniform convergence on members of A.

14.4 SUMMARY

We study countable space and its properties. We study covering space and uniform space. We study some examples of covering space. We study function space.

14.5 KEYWORD

COUNTABLE : that can form a plural or be used with the indefinite article

COMPACT : Closely and neatly packed together; dense

CAYLEY : A pirate that scours the seven seas. If you are unlucky enough to meet a Cayley on the ocean, expect to be in Davy Jones locker within a matter of moments.

14.6 QUESTIONS FOR REVIEW

1 A uniform space of countable type has a basis of (countable) uniformly locally finite uniform coverings.

3 Let X be a completely regular space.

1. The completion of X in the uniformity C(X) is (υX, C(υX)).
2. The completion of X in the uniformity C (X) is (βX, C(βX)).

4 1) A closed subset of a complete uniform space is complete.

2) A complete subspace of a Hausdorff uniform space is closed.

5 The uniform completion of a (Hausdorff) uniform space is itself a (Hausdorff) uniform space.

1 Let X1, X2 be complete Hausdorff uniform space, and let A1, A2 be dense subsets of X1, X2 respectively. If A1 and A2 are uniformly equivalent then so are X1 and X2.

2 Every uniform space is uniformly isomorphic to a dense subspace of a complete uniform space. Each Hausdorff uniform space is uniformly isomorphic to a dense subspace of a complete Hausdorff uniform space.

3 A uniform space is compact iff it is complete and totally bounded.

14.7 SUGGESTION READING AND REFERENCES

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14.8 ANSWER TO CHECK YOUR

PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1.4

Q 2 Check in Section 1

Check in Progress-II

Answer Q. 1 Check in Section 2

Q 2 Check in Section 2